Solving Problems Involving Cost, Revenue, Profit

The **cost function C(x)** is the total cost of making x items. If the cost per item is fixed, it is equal to the cost per item (c) times the number of items produced (x), or $C(x) = c \cdot x$.

The **price function p(x)** – also called the demand function – describes how price affects the number of items sold. Normally, when the price increases, customers will not demand as many items, and so x will decrease. To sell more items, the price usually has to decrease.

The **revenue function R(x)** is the income from sales. It is equal to the price times the number sold, or $R(x) = x \cdot p(x)$.

The **profit function P(x)** is the money that is left over from the revenue (income) after the costs (expenses) have been subtracted. In other words, $P(x) = R(x) – C(x)$.

The **break-even point** occurs when the total revenue equals the total cost - or, in other words, when the profit is zero. To solve for a break-even quantity, set $P(x) = 0$ and solve for x using factored form or the quadratic formula.

Max and Min Problems

Max and min problems can be solved using any of the forms of quadratic equation:

Vertex form	$y = a(x - h)^2 + k$	the vertex is (h, k)
Factored form	$y = a(x - p)(x - q)$	the vertex will lie on the axis of symmetry, which is found using the mean of the roots p and q.
Standard form	$y = ax^2 + bx + c$	find the vertex by completing the square (or by using the shortcut method related to the axis of symmetry)

A vertex is a **minimum** if the parabola opens up ($a > 0$)
A vertex is a **maximum** if the parabola opens down ($a < 0$)
Depending on the situation, the "best" solution (either max or min) is called the **optimal value**.

Warm up:

You are the sole owner of a denim store downtown, Toronto. Last week, you sold 200 pairs of jeans priced at $36. You buy the pants from a local manufacturer located in Montreal, Quebec. Calculate your **total revenue** for the last week.

Revenue = (# of jeans sold)(unit price)
= (200)(36)
= $7200

The Real Deal: How to maximize the Revenue

Ex.1 Ever since you took grade 11 math course in high school with Mr. Bulut, you have been wondering if you can apply **maximizing total revenue** concept in your business to make more money. When you set your price for $36, you sell average of 200 pairs a week. After doing a mini survey, you find out that for each $2 increase in price, the demand for jeans is less and 5 fewer pants are sold.

a) What price will **maximize** your total **revenue**?

Revenue = (# of items sold)(Unit Price)

		# of times of price increase
	= (200)(36)	
	(200−5)(36+2)	1 time
	(200−10)(36+4)	2
	(200−15)(36+6)	3
R(x) =	(200−5x)(36+2x)	x

$R(x) = (200-5x)(36+2x)$
or
$R(x) = 7200 + 400x - 180x - 10x^2$
$R(x) = -10x^2 + 220x + 7200$

$R(x) = -10x^2 + 220x + 7200$ "x" represents the number of times of price increase

By completing the square you can find the vertex which will give you the max revenue

$\frac{-22}{2} = -11$

$(-11)^2 = 121$

$= -10(x^2 - 22x) + 7200$
$= -10(x^2 - 22x + 121 - 121) + 7200$
$= -10(x^2 - 22x + 121) + 1210 + 7200$
$R(x) = -10(x-11)^2 + 8410$

V(11, 8410)

if you increase the price 11 times

your revenue will max at $8410

# of jeans sold = 200 − 5x
= 200 − 5·11
= 145 jeans sold

Unit (Price) p(x) = 36 + 2x
= 36 + 2·11
= $58

∴ When the price is set to $58, I will maximize my revenue!

Ex.2 When priced at $40 each, a toy company sells 5000 toys. The manufacturer estimates that each $1 increase in price will decrease sales by 100 units. Find the unit price of a toy that will maximize the total revenue.

Revenue = (# of toys sold)(Unit price)

$R(x) = (5000 - 100x)(40 + x)$ expand by FOIL

$= 20000 + 5000x - 4000x - 100x^2$

$R(x) = -100x^2 + 1000x + 20000$ complete the square

$= -100(x^2 - 10x) + 20000$ $\frac{-10}{2} = -5$ $(-5)^2 = 25$

$= -100(x^2 - 10x + 25 - 25) + 20000$

$= -100(x^2 - 10x + 25) + 2500 + 20000$

$= -100(x - 5)^2 + 22500$

$V(5, 22500)$

# of toys sold $= 5000 - 100x$
$= 5000 - 100(5)$
$= \underline{4500}$

Unit Price $p(x) = 40 + x$
$= 40 + 5$
$= \underline{\underline{\$45}}$

∴ When the price is set to $45, the toy company will reach its max revenue ($202,500).

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PRACTICE QUESTIONS

1. The city transit system carries 24800 bus riders per day for a fare of \$1.85. The city hopes to reduce car pollution by getting more people to ride the bus, while maximizing the transit system's revenue at the same time. A survey indicates that the number of riders will increase by 800 for every \$0.05 decrease in the fare. What fare will produce the **greatest revenue**?

2. The Thunderbirds professional indoor soccer team has 900 season ticket holders. The management of the team wants to increase the current price of \$400. A survey indicated that for every \$20 increase in price, the team will lose 15 season ticket holders. What price would maximize revenue from season ticket holders? What is the **maximum revenue** the team could receive?

3. The school council sells sweatshirts to raise funds. The students sell 500 sweatshirts a year at \$45 each. They are planning to decrease the price to generate more sales. An informal survey was taken showing that for every \$1 decrease in price they can expect to sell an additional 20 sweatshirts. If the survey results are correct, what price would maximize revenue from sweatshirt sales? How many sweatshirts must be sold? What would be the **maximum revenue** generated?

4. The path of a golf ball can be modelled by the function $h = -0.003d^2 + 0.6d$, where *h* is the height of the golf ball, in metres, and *d* is the horizontal distance travelled, in metres. What is the maximum height of the golf ball? At what horizontal distance does the golf ball reach its maximum height?

5. A ball is thrown vertically upward off the roof of a 34 m tall building. The height of the ball *h* in metres, can be approximated by the function $h = -5t^2 + 10t + 34$ where *t* is the time in seconds, after the ball is thrown.

   a) Sketch the graph.

   b) How high is the ball after 2 s?

   c) Find the maximum height of the ball.

6 A tennis ball is thrown up into the air. Its height *h* in metres after *t* seconds, is given by the function $h = -4.9t^2 + 19.6t + 2.1$

   a) Sketch the graph.

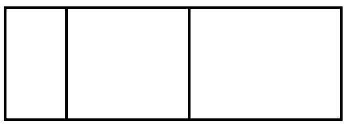
   b) Determine the maximum height of the ball and the time it takes to reach it.

   c) How high is the ball after 3 s?

7. A rectangular lot is bordered on one side by a stream and on the other three sides by 600 metres of fencing. Determine the dimensions of the lot if its area is a maximum.

8. A lifeguard marks off a rectangular swimming area at a beach with 200 m of rope. What is the greatest area of water she can enclose if the rope only makes 3 sides of the rectangle? (No rope is needed along the shore side of the swimming area.)

9. A rectangular field is enclosed by a fence and divided into three parts by another section of fence which runs parallel to the short sides. If the 600 m of fence used encloses a maximum area, what are the dimensions of the field?



10 Determine the maximum possible area for a rectangle with perimeter 20

14. An amusement park charges \$8 admission and averages 2000 visitors per day. A survey shows that for each \$1 increase in the admission price, 100 fewer people would visit the park.

   a) Determine what price the amusement park should charge to maximize **revenue**.
   b) At what price(s) will the revenue be equal to \$0?
   c) Find the price(s) that would generate revenue of \$11500.

15. Vitaly and Jen have 24 m of fencing to enclose a vegetable garden at the back of their house. What are the dimensions of the largest rectangular garden they could enclose with this length of fencing?

1. The city transit system carries 24800 bus riders per day for a fare of \$1.85. The city hopes to reduce car pollution by getting more people to ride the bus, while maximizing the transit system's revenue at the same time. A survey indicates that the number of riders will increase by 800 for every \$0.05 decrease in the fare. What fare will produce the **greatest revenue**?

Revenue = (# of bus riders)(bus fare)

$R(x) = (24800 + 800x)(1.85 - 0.05x)$

$= 45880 - 1240x + 1480x - 40x^2$

$R(x) = -40x^2 + 240x + 45880$ complete the square

$= -40(x^2 - 6x) + 45880$ $\frac{-6}{2} = -3$ $(-3)^2 = 9$

$= -40(x^2 - 6x + 9 - 9) + 45880$

$= -40(x^2 - 6x + 9) + 360 + 45880$

$= -40(x-3)^2 + 46240$

$V(3, 46240)$

Unit Price $\cdot$ $p(x) = 1.85 - 0.05x$

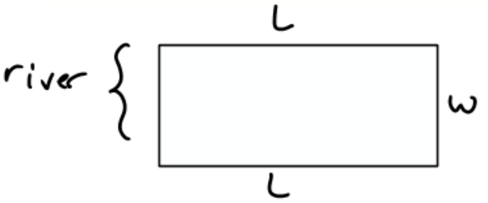
$= 1.85 - 0.05(3)$

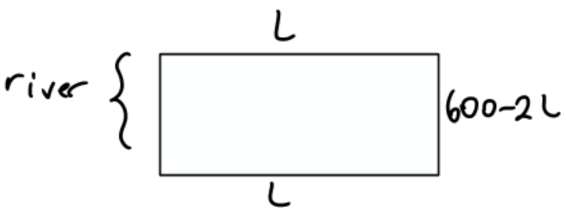
$= 1.85 - 0.15$

$= \$1.70$

$\therefore$ When the bus fare is \$1.70, the max revenue is \$46240

7. A rectangular lot is bordered on one side by a stream and on the other three sides by 600 metres of fencing. Determine the dimensions of the lot if its area is a maximum.





$l + l + w = 600$

$2l + w = 600$ if $2l + w = 600$, then $w = 600 - 2l$

Area $= l \times w$

$= l(600 - 2l)$

$= 600l - 2l^2$

$= -2l^2 + 600l$

$= -2(l^2 - 300l)$ $-\frac{300}{2} = -150$ $(-150)^2 = 22500$

$= -2(l^2 - 300l + 22500 - 22500)$

$= -2(l^2 - 300l + 22500) + 45000$

$= -2(l - 150)^2 + 45000$

$V(150, 45000)$ $\therefore l = 150$

$w = 300$

4. The path of a golf ball can be modelled by the function $h = -0.003d^2 + 0.6d$, where h is the height of the golf ball, in metres, and d is the horizontal distance travelled, in metres. What is the maximum height of the golf ball? At what horizontal distance does the golf ball reach its maximum height?

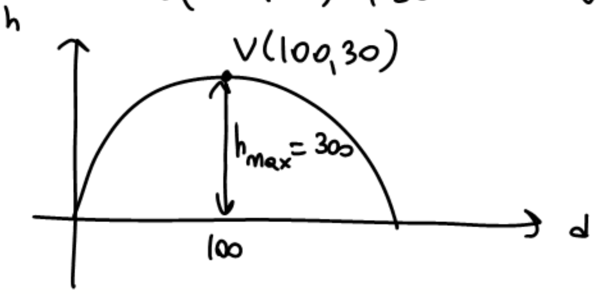
$h = -0.003d^2 + 0.6d$

$= -0.003(d^2 - 200d)$ $\rightarrow -\frac{200}{2} = -100$ $(-100)^2 = 10000$

$= -0.003(d^2 - 200d + 10000 - 10000)$

$= -0.003(d^2 - 200d + 10000) + 30$

$= -0.003(d - 100)^2 + 30$ $V(100, 30)$



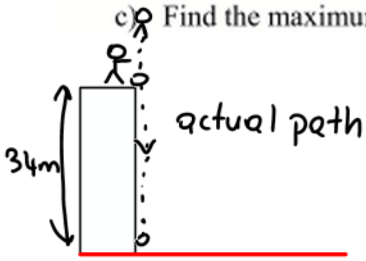
When the ball is at 100m horizontal distance, it reache its max height of 30m.

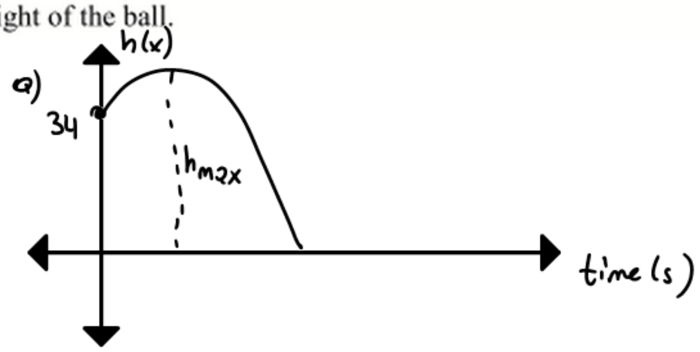
5. A ball is thrown vertically upward off the roof of a 34 m tall building. The height of the ball h in metres, can be approximated by the function $h = -5t^2 + 10t + 34$ where t is the time in seconds, after the ball is thrown.

a) Sketch the graph.

b) How high is the ball after 2 s?

c) Find the maximum height of the ball.





b) $h(t) = -5t^2 + 10t + 34$

$h(2) = -5(2)^2 + 10(2) + 34$

$= -20 + 20 + 34$

$h(2) = 34m$

c) $h(t) = -5t^2 + 10t + 34$

$= -5(t^2 - 2t) + 34$ $\rightarrow -\frac{2}{2} = -1$ $(-1)^2 = 1$

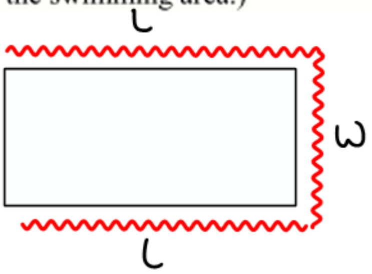
$= -5(t^2 - 2t + 1 - 1) + 34$

$= -5(t^2 - 2t + 1) + 5 + 34$

$= -5(t - 1)^2 + 39$ $V(1, 39)$

$\underline{h_{max} = 39m}$

8 A lifeguard marks off a rectangular swimming area at a beach with 200 m of rope. What is the greatest area of water she can enclose if the rope only makes 3 sides of the rectangle? (No rope is needed along the shore side of the swimming area.)



$2l + w = 200$ then $w = 200 - 2l$

$$\begin{aligned} \text{Area} &= l \cdot w \\ &= l(200 - 2l) \\ &= 200l - 2l^2 \end{aligned}$$

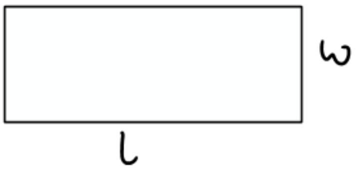
$$\begin{aligned} A(l) &= -2l^2 + 200l \\ &= -2(l^2 - 100l) \qquad \frac{-100}{2} = -50 \quad (-50)^2 = 2500 \\ &= -2(l^2 - 100l + 2500 - 2500) \\ &= -2(l - 50)^2 + 5000 \end{aligned}$$

$V(50, 5000)$
length area

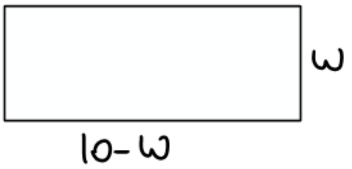
$$\begin{aligned} \text{width} &= 200 - 2l \\ &= 200 - 2(50) \\ \text{width} &= 100 \end{aligned}$$

$\therefore$ length = 50
width = 100

10 . Determine the maximum possible area for a rectangle with perimeter 20 .



$$\begin{aligned} P &= 2(l + w) \\ 20 &= 2(l + w) \\ 10 &= l + w, \text{ then } l = 10 - w \end{aligned}$$



$$\begin{aligned} A &= (10 - w)w \\ &= 10w - w^2 \\ &= -w^2 + 10w \\ &= -(w^2 - 10w) \qquad \frac{-10}{2} = -5 \quad (-5)^2 = 25 \\ &= -(w^2 - 10w + 25 - 25) \\ &= -(w - 5)^2 + 25 \end{aligned}$$

$V(5, 25)$ length = 5, width = 5

$\therefore$ The max area occurs when length is 5m and width is 5m.

2. The Thunderbirds professional indoor soccer team has 900 season ticket holders. The management of the team wants to increase the current price of $400. A survey indicated that for every $20 increase in price, the team will lose 15 season ticket holders. What price would maximize revenue from season ticket holders? What is the **maximum revenue** the team could receive?

$R(x) = (900-15x)(400+20x)$ Let "x" be the # of increase in price.

$= 360000 + 18000x - 6000x - 300x^2$

$= -300x^2 + 12000x + 360000$

$= -300(x^2 - 40x) + 360000$ $-\frac{40}{2} = -20$ $(-20)^2 = 400$

$= -300(x^2 - 40x + 400 - 400) + 360000$

$= -300(x-20)^2 + 120000 + 360000$

$= -300(x-20)^2 + 480,000$

$V(20, 480000)$

$\therefore$ $480,000

3. The school council sells sweatshirts to raise funds. The students sell 500 sweatshirts a year at $45 each. They are planning to decrease the price to generate more sales. An informal survey was taken showing that for every $1 decrease in price they can expect to sell an additional 20 sweatshirts. If the survey results are correct, what price would maximize revenue from sweatshirt sales? How many sweatshirts must be sold? What would be the **maximum revenue** generated?

$R(x) = (500+20x)(45-x)$

$= 22500 - 500x + 900x - 20x^2$

$= -20x^2 + 400x + 22500$

$= -20(x^2 - 20x) + 22500$ $-\frac{20}{2} = -10$ $(-10)^2 = 100$

$= -20(x^2 - 20x + 100 - 100) + 22500$

$= -20(x-10)^2 + 2000 + 22500$

$R(x) = -20(x-10)^2 + 24500$

$V(10, 24500)$

Price $= 45 - x$
$= 45 - 10$
$= \$35$

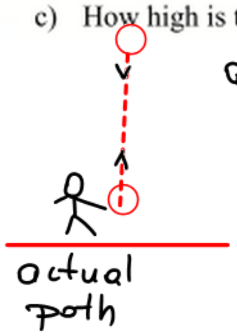
$\therefore$ When the ticket price is set to $35, the max revenue will be $24,500.

6 A tennis ball is thrown up into the air. Its height h in metres after t seconds, is given by the function $h = -4.9t^2 + 19.6t + 2.1$

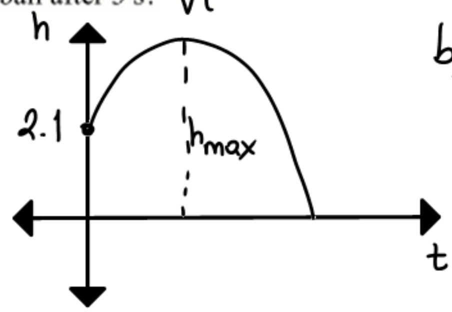
a) Sketch the graph.

b) Determine the maximum height of the ball and the time it takes to reach it.

c) How high is the ball after 3 s? V(



a)



b) $h = -4.9t^2 + 19.6t + 2.1$

$= -4.9(t^2 - 4t) + 2.1$ $\left(\frac{-4}{2}\right) = -2$

$= -4.9(t^2 - 4t + 4 - 4) + 2.1$ $(-2)^2 = 4$

$= -4.9(t-2)^2 + 19.6 + 2.1$

$h = -4.9(t-2)^2 + 21.7$

$V(2, 21.7)$

$\therefore$ It takes 2 seconds to reach max height of 21.7 m.

c) $h(3) = -4.9(3-2)^2 + 21.7$

$= -4.9(1)^2 + 21.7$

$= -4.9 + 21.7$

$\boxed{h(3) = 16.8}$

14. An amusement park charges \$8 admission and averages 2000 visitors per day. A survey shows that for each \$1 increase in the admission price, 100 fewer people would visit the park.

a) Determine what price the amusement park should charge to <u>maximize</u> **revenue**.

b) At what price(s) will the revenue be equal to \$0?

c) . Find the price(s) that would generate revenue of \$11500.

Revenue = (# of visitors)(park fee)

$R(x) = (2000 - 100x)(8 + x)$ Let "x" be the number of times price gets increased. Expand.

$R(x) = 16000 + 2000x - 800x - 100x^2$

$= -100x^2 + 1200x + 16000$ complete the square

$= -100(x^2 - 12x) + 16000$ $\frac{-12}{2} = -6$ $(-6)^2 = 36$

$= -100(x^2 - 12x + 36 - 36) + 16000$

$= -100(x-6)^2 + 3600 + 16000$

$R(x) = -100(x-6)^2 + 19600$

$V(6, 19{,}600)$

Price $= p(x) = 8 + x$

$= 8 + 6$

$= \$14$

$\therefore$ The fee should be \$14 to max the revenue.

b) $R(x) = 0$

$-100x^2 + 1200x + 16000 = 0$

$\frac{-\cancel{100}(x^2 - 12x - 160)}{\cancel{-100}} = \frac{0}{-100}$

$x^2 - 12x - 160 = 0$

$(x + 8)(x - 20) = 0$

1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 80, 160

$x + 8 = 0$ $\quad$ $x - 20 = 0$

$x = -8$ $\quad$ $x = 20$

$p(x) = 8 + x$
$p(-8) = 8 + (-8)$
$p(-8) = 0$
$\$0$

$p(x) = 8 + x$
$p(20) = 8 + 20$
$p(20) = 28$
$\$28$

∴ When the price is set for \$0 and \$28, the revenue is \$0.