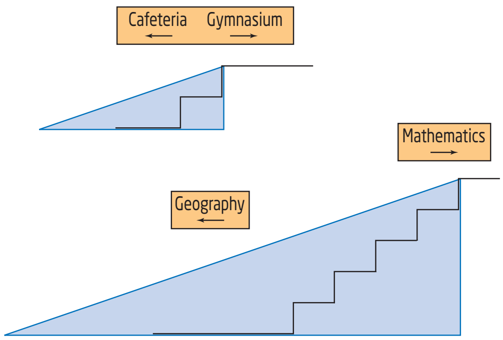
7.3 The Tangent Ratio

Cory is installing wheelchair ramps at a high school. Not all locations require the same vertical climb, so he will need to adjust the length of the ramp in each case. In general, a wheelchair ramp should have a slope of not more than $\frac{1}{12}$.

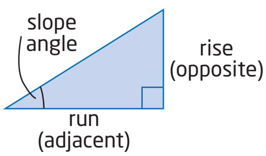
How can Cory ensure that all ramps have the same slope? How is the slope related to the angle the ramp makes with the floor?



Investigate

Tools
- grid paper
- protractor
- ruler

slope angle
- angle opposite the rise and adjacent to the run



What is the tangent ratio and how is it related to slope?

Method 1: Use Pencil and Paper

1. Draw three similar right triangles on grid paper for each **slope angle**. Each triangle represents a different ramp.
- 10°
- 25°
- 60°

2. a) Measure the rise and run of each triangle, and use these to calculate the slope. Record your results in a table like this one.

Triangle	Slope Angle	Rise	Run	Slope (to three decimal places)

b) What do you notice about the slope of the ramp for similar right triangles?

3. The slope in your table in step 2 is called the **tangent** of the slope angle.

   a) Draw three more similar ramps with a different slope angle than those in step 1. Measure the rise, run, and slope of these ramps and record your results in your table. Explain what you notice.

   b) Add a column to your table labelled **Tangent** and use a calculator to calculate the tangent ratio of the slope angle. What do you notice?

4. How can you find the tangent of the other acute angle in the triangle? Rotate each original triangle 90° counterclockwise and repeat the investigation.

5. **Reflect**

   a) What is the tangent ratio of an angle? What is it the same as?

   b) How does the tangent of an angle change when you change the size of the triangle but keep the angle the same?

Method 2: Use *The Geometer's Sketchpad*®

1. Open *The Geometer's Sketchpad*® and begin a new sketch.

2. Construct a small ramp.
   - Construct a short horizontal line segment.
   - Select the segment and the right endpoint.
   - From the **Construct** menu, choose **Perpendicular Line**.
   - From the **Construct** menu, choose **Point on Perpendicular Line**.
   - Select the perpendicular line. From the **Display** menu, choose **Hide Perpendicular Line**.
   - Construct segments to complete the right triangle.

3. a) Set the precision of the measurements to thousandths. From the **Edit** menu, choose **Preferences**. Set the **Precision** to **thousandths** for **Angle**, **Distance**, and **Other**.

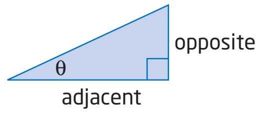
   b) Measure the rise and run of the ramp, and use these to calculate the slope.
   - Select the vertical and horizontal segments. From the **Measure** menu, choose **Length**.
   - From the **Measure** menu, choose **Calculate**. A calculator will appear.
   - Select the rise measure.
   - Click on ÷.
   - Select the run measure.
   - Click **OK**.

   c) A ratio measure will appear. What does this ratio represent?

   d) Right click on the measure and choose **Label Measurement** to rename the measure to correspond to your answer to part c).

tangent of an angle

- the ratio of the side opposite an angle to the side adjacent to the angle



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Literacy Connections

The short form for tangent is tan. The Greek letter θ, pronounced "theta," is often used to represent angles.

Tools

- computer with *The Geometer's Sketchpad*®

Technology Tip

- Holding the **Shift** key as you draw a segment keeps it horizontal or vertical.
- From the **Display** menu, use the **Hide** command, or press **Ctrl + H**, to tidy up your sketch.

4. Measure the angle the ramp makes with the floor.
   - Select the three points in order so that you select the point where the ramp meets the floor second.
   - From the **Measure** menu, choose **Angle**.

   Label this measure the **slope angle**.

Technology Tip

In step 3, you created a dynamic calculation that automatically updated itself when you changed your sketch.

5. a) Click and drag each vertex of the triangle. Explain what happens in each case.
   b) Does the triangle remain a right triangle? Explain why or why not.
   c) Return the triangle to the way you had it at the end of step 3.

6. a) Record how the measures change as you change the triangle. To make a table in *The Geometer's Sketchpad*®:
      - Select the measurements in the following order: slope angle, rise, run, slope.
      - From the **Graph** menu, choose **Tabulate**.
      - Right click on the table and choose **Add Table Data**.
      - Choose **Add 10 Entries As Values Change**.

      Click and drag the vertex above the right angle to change the slope angle. Ten sets of measurements will be recorded in your table.
   b) Discover what happens to the slope if you change the size of the triangle but keep the slope angle. Make a new table, as in part a). Click on and drag the vertex at the right angle to make similar triangles with the same slope angle. Record your results in your new table. Describe what happens to each of the following measures:
      - rise
      - run
      - slope
      - slope angle

7. Repeat step 6 for a different slope angle. Does the slope of the ramp change if the angle remains the same? Explain.

8. The slope in your table in steps 6 and 7 is called the **tangent** of the slope angle.

   You can use *The Geometer's Sketchpad*® to calculate the tangent of an angle.
   - From the **Measure** menu, choose **Calculate**.
   - Click on the **Functions** key. From the menu of functions, click on **tan**.
   - Select the angle measure of the ramp.
   - Click **OK**.

   Calculate the tangents of several slope angles and compare them to the slope of the ramp. What do you notice?

9. How can you find the tangent of the other acute angle in the triangle? Rotate your original triangle 90° counterclockwise and repeat the investigation.

10. **Reflect**

  a) What is the tangent ratio of an angle? What is it the same as?

  b) How does the tangent of an angle change when you change the size of the triangle but keep the angle the same?

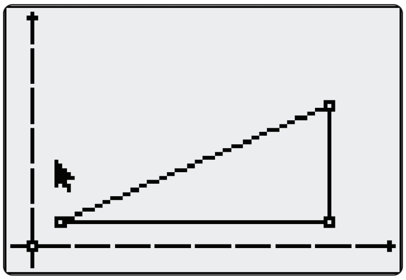
Method 3: Use a Graphing Calculator

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

1. Press APPS and choose **Cabri Jr**. If the axes are not visible, choose the **F5** menu. Choose **Hide/Show**, and then **Axes**. Move the cursor to the origin. Press ALPHA and move the origin until the first quadrant occupies most of the screen, as shown.

2. Construct a small ramp.

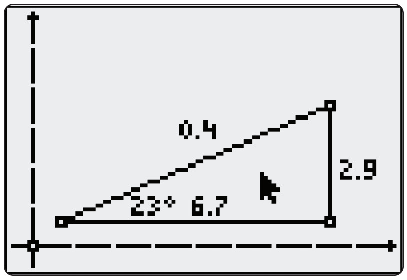


  - Choose the **Segment** tool from the **F2** menu. Draw a horizontal line segment, as shown.
  - Choose **Perp.** from the **F3** menu.
  - Move the cursor to the right endpoint of the segment. Press ENTER. Move the cursor to the segment until it flashes. Press ENTER. A line perpendicular to the segment appears, through the right endpoint.
  - Choose **Point** from the **F2** menu, and then **Point on**. Move the cursor to a point on the perpendicular line, and press ENTER.
  - Choose **Hide/Show** from the **F5** menu. Then, choose **Object**. Move the cursor to the perpendicular line until it flashes. Press ENTER. The line is hidden.
  - Construct segments to complete the right triangle.

3. a) Measure the rise and run of the ramp, and use these to calculate the slope.
  - Choose **Measure** from the **F5** menu. Then, choose **D. & Length**.
  - Move the cursor to the run until it flashes. Press ENTER. Move the measurement to a suitable location, and press ENTER again.
  - Repeat this procedure to measure the rise.
  - Choose **Calculate** from the **F5** menu. Move the cursor to the rise measurement, and press ENTER. Move the cursor to the run measurement. Press ENTER. Then, press ÷. Move the calculation to a convenient location, and press ENTER.

  b) What does this ratio represent?

4. Measure the angle the ramp makes with the floor.
   - Choose **Measure** from the **F5** menu. Then, choose **Angle**.
   - Move the cursor to the top vertex of the triangle. Press ENTER. Move to the left vertex. Press ENTER. Move to the third vertex. Press ENTER. Move the measurement to a convenient location, and press ENTER.



   Call this the **slope angle**.

5. a) Move the cursor to one of the vertices of the triangle. Press ALPHA. Drag the vertex of the triangle. Explain what happens. Try dragging the other vertices, one at a time. Explain what happens.
   b) Does the triangle remain a right triangle? Explain why or why not.
   c) Return the triangle to the way you had it at the end of step 4.

6. a) Select and drag the vertex above the right angle to change the slope angle. Record the measures in a table like this one.

Slope Angle	Rise	Run	Slope (to three decimal places)

   b) Discover what happens to the slope if you change the size of the triangle but keep the slope angle the same. Select and drag the left vertex of the triangle. Record your results in the table from part a). Describe what happens to each of the following measures:
   - rise
   - run
   - slope
   - slope angle

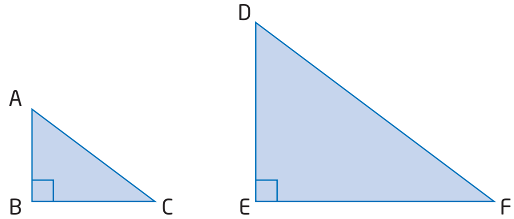
7. Repeat step 6 for a different slope angle. Does the slope of the ramp change if the angle remains the same? Explain.

8. The slope in your table in steps 6 and 7 is called the **tangent** of the slope angle. Calculate the tangent of several slope angles and compare it to the slope of the ramp. What do you notice?

9. How can you find the tangent of the other acute angle in the triangle? Rotate your original triangle 90° counterclockwise and repeat the investigation.

10. **Reflect**
   a) What is the tangent ratio of an angle? What is it the same as?
   b) How does the tangent of an angle change when you change the size of the triangle but keep the angle the same?

In the Investigate, you saw that the tangent of an acute angle in a right triangle is constant if the angle stays the same, even if the size of the triangle changes. This is because of the properties of similar triangles. Since the ratios of corresponding sides of similar triangles are equal, and the tangent is a ratio of sides, the tangents of the angles of similar triangles are equal. For example, in similar triangles △ABC and △DEF,



$$\tan C = \frac{\text{opposite}}{\text{adjacent}} \quad \text{and} \quad \tan F = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{AB}{BC} \qquad\qquad = \frac{DE}{EF}$$

By similar triangles,

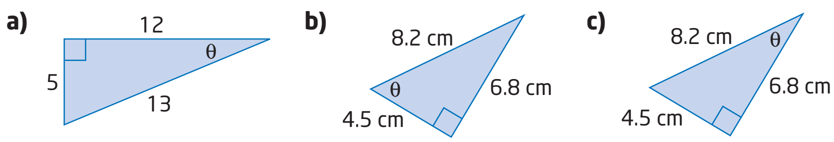
$$\frac{AB}{DE} = \frac{BC}{EF}$$

$$\frac{AB}{BC} = \frac{DE}{EF}$$ **Divide both sides by BC and multiply both sides by DE.**

So, tan C = tan F.

Example 1 Find the Tangent Ratio From Given Sides

Find tan θ for each triangle, expressed as a fraction and as a decimal correct to four decimal places.



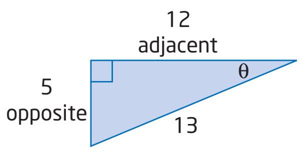
Solution

a) Identify the opposite and adjacent sides relative to the angle θ.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{5}{12}$$

$$\doteq 0.4167$$

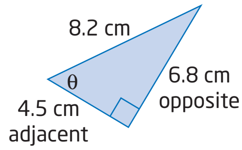


Literacy Connections

Expressions involving tangent mean "the tangent of." For example, tan A means "the tangent of angle A." Similarly, tan 25° means "the tangent of 25°."

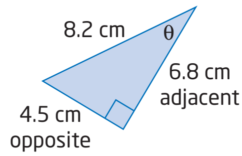
b) Identify the opposite and adjacent sides.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{6.8}{4.5}$$
$$\doteq 1.5111$$



c) When you go from one acute angle to the other in a right triangle, the opposite and adjacent sides relative to the angle become reversed.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{4.5}{6.8}$$
$$\doteq 0.6618$$



This is the same triangle as in part b), but I am calculating the tangent of the other acute angle.

Technology Tip

Not all scientific calculators work the same way. With some, you press
TAN 25 =

With others, you press
25 TAN

On a graphing calculator, press
TAN 25) =

Make sure your calculator is in degree mode.

Example 2 Find the Tangent of an Angle

Evaluate each of the following with a calculator. Record your answer rounded to four decimal places.

a) tan 25°

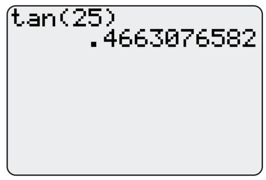
b) tan 60°

Solution

Use a scientific or graphing calculator.

a) tan 25° ≐ 0.4663

b) tan 60° ≐ 1.7321



Calculators have an inverse function that allows you to apply the tangent ratio in reverse. If you know the ratio $\frac{\text{opposite}}{\text{adjacent}}$, you can find the angle whose tangent this ratio represents.

Example 3 Find an Angle Using the Tangent Ratio

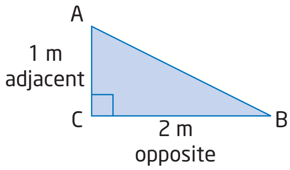
Fiona is building a skateboarding ramp. She wants the ramp to rise 1 m in a horizontal distance of 2 m. At what acute angles should she cut the wood, rounded to the nearest degree?

Solution

Sketch and label a diagram of the ramp.

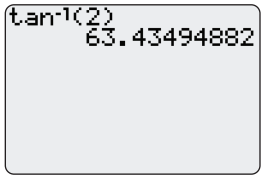
Find $\angle A$.

$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{2}{1}$$
$$= 2$$



To find $\angle A$, calculate the inverse tangent of 2.

$$\angle A = \tan^{-1}(2)$$
$$\doteq 63.43$$

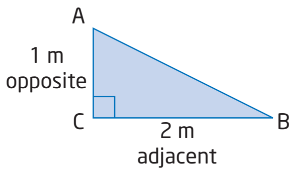


One of the acute angles is about 63°.

Find $\angle B$.

Method 1: Apply the Tangent Ratio

$$\tan B = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{1}{2}$$
$$= 0.5$$



Apply the inverse tangent operation.

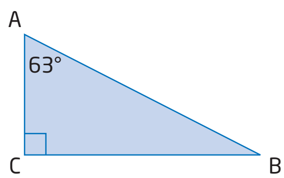
$$\angle B = \tan^{-1}(0.5)$$
$$\doteq 26.56$$

The other acute angle is about 27°.

Method 2: Use Geometric Reasoning

The sum of three interior angles in any triangle is 180°.

$$\angle A + \angle B + \angle C = 180°$$
$$63° + \angle B + 90° = 180°$$
$$\angle B = 180° - (63° + 90°)$$
$$= 180° - 153°$$
$$= 27°$$



The other acute angle is about 27°.

Fiona should cut the wood using acute angles of approximately 63° and 27°.

Note that Method 1 will give a more accurate measure for the angle, since only given information is being used.

Literacy Connections

Tangent and **inverse tangent** are opposite operations, like addition and subtraction. For example,

$\tan 60° \doteq 1.7321$

$\tan^{-1}(1.7321) \doteq 60°$

The second statement is read as "the inverse tangent of 1.7321 is approximately equal to 60 degrees."

Technology Tip

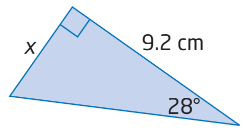
With some scientific calculators, you press **2nd [TAN^{-1}] 0.5 =**

With others, you press **0.5 2nd [TAN^{-1}]**

The tangent ratio relates two sides of a right triangle and an angle. If you know an angle and the length of one of the legs of the triangle, you can find the length of the other leg.

Example 4 Find a Side Length Using the Tangent Ratio

Find the length, x, in the diagram, rounded to the nearest tenth of a centimetre.



Solution

Write the tangent ratio for the given angle in terms of the side lengths.

$$\tan 28° = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 28° = \frac{x}{9.2}$$

$$9.2(\tan 28°) = x$$ Multiply both sides by 9.2.

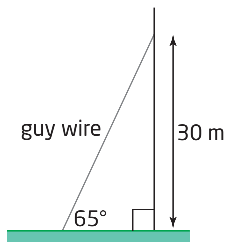
$$4.891 \doteq x$$ 9.2 × 28 TAN = or 9.2 × TAN 28 =

The length of side x is about 4.9 cm.

Example 5 Solve a Multi-Step Problem Using the Tangent Ratio

A radio transmitter is to be supported with a guy wire, as shown. The wire is to form a 65° angle with the ground and reach 30 m up the transmitter.

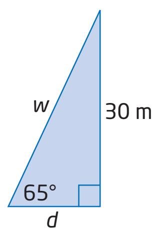
The wire can be ordered in whole-number lengths of metres. How much wire should be ordered?



Solution

Use the tangent ratio of the given angle to find the distance, d, from the tower that the guy wire should be secured.

Then, apply the Pythagorean theorem to find the length, w, of wire needed.



$$\tan 65° = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 65° = \frac{30}{d}$$

$$d(\tan 65°) = 30$$ Multiply both sides by d.

$$d = \frac{30}{\tan 65°}$$ Divide both sides by tan 65°.

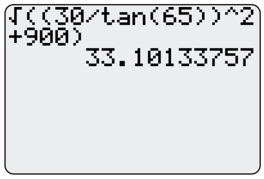
Apply the Pythagorean theorem to find the length of wire needed.

$w^2 = d^2 + 30^2$

$w^2 = \left(\dfrac{30}{\tan 65°}\right)^2 + 900$

$w = \sqrt{\left(\dfrac{30}{\tan 65°}\right)^2 + 900}$

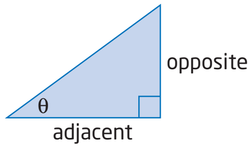
$w \doteq 33.101$



The length of guy wire needed is about 33.1 m. It can only be ordered in whole-number lengths. 33 m is too short, so round up to the next metre. At least 34 m of guy wire should be ordered.

Key Concepts

- The ratio of the opposite side to the adjacent side of an angle in a right triangle is called the tangent of that angle.



$\tan \theta = \dfrac{\text{opposite}}{\text{adjacent}}$

- You can use a scientific or graphing calculator to
  – express the tangent of an angle as a decimal
  – find one of the acute angles when both leg lengths are known in a right triangle
  – find a side length if one acute angle and one leg of a right triangle are known

Communicate Your Understanding

C1 Explain how the tangent of the slope angle is related to the slope of a ramp.

C2 **a)** Explain how you can identify the opposite and adjacent sides of a right triangle.

b) Explain how these can change in a given triangle. Use a diagram to support your explanation.

C3 **a)** Explain what each calculator function does:
- tangent
- inverse tangent

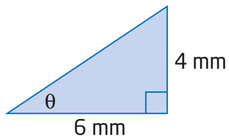
b) When would you use each function?

c) How are these functions related to each other?

Practise

For help with questions 1 and 2, see Example 1.

1. Find the tangent of the angle indicated, to four decimal places.

a)


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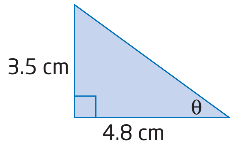

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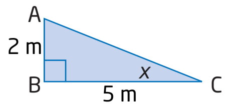

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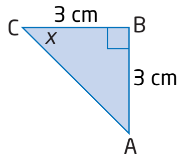

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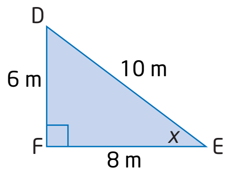

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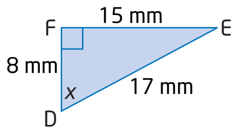


2. Refer to question 1. Find the tangent of the other acute angle, to four decimal places.

For help with question 3, see Example 2.

3. Evaluate with a calculator. Record your answer to four decimal places.

a) tan 65° **b)** tan 15°
c) tan 62° **d)** tan 5°
e) tan 30.7° **f)** tan 82.4°
g) tan 20.5° **h)** tan 45°

For help with questions 4 to 8, see Example 3.

4. Find the measure of each angle, to the nearest degree.

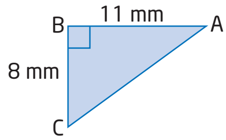
a) $\tan \theta = 1.5$ **b)** $\tan A = \frac{3}{4}$

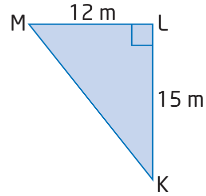
c) $\tan B = 0.6000$ **d)** $\tan W = \frac{4}{5}$

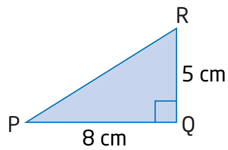
e) $\tan C = 0.8333$ **f)** $\tan \theta = \frac{6}{7}$

g) $\tan X = 3.0250$ **h)** $\tan \theta = \frac{15}{9}$

5. Find the measures of both acute angles in each triangle, to the nearest degree.

a)


b)


c)


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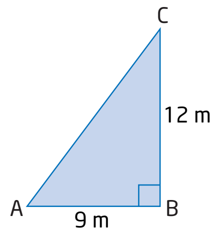
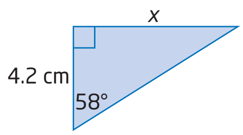
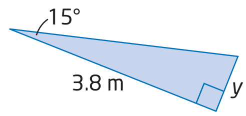
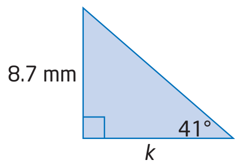
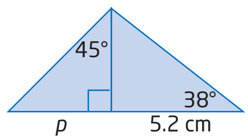


6. Find the length of the unknown side, to the nearest tenth.

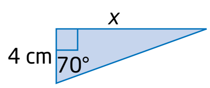
a)


b)


c)


d)


7. Find the length of x, to the nearest tenth of a metre.

a)


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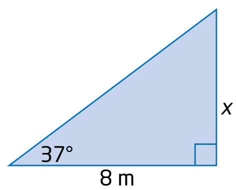
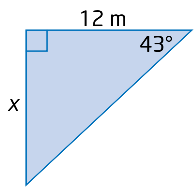
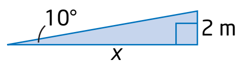
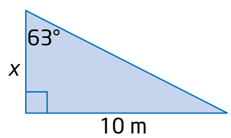
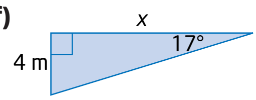


c)


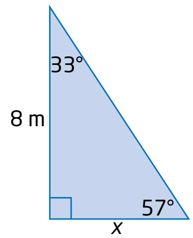
d)


e)


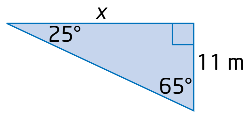
f)


8. Find the length of x, to the nearest tenth of a metre.

a)

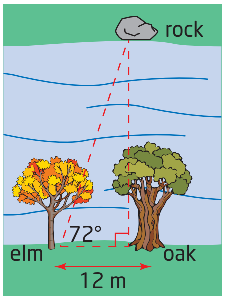


b)



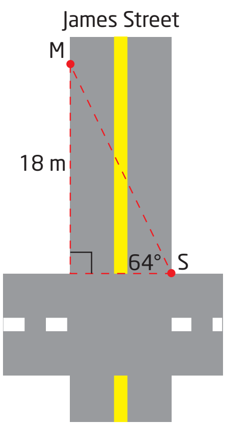
Connect and Apply

9. To measure the width of a river, Kirstyn uses a large rock, an oak tree, and an elm tree, which are positioned as shown.

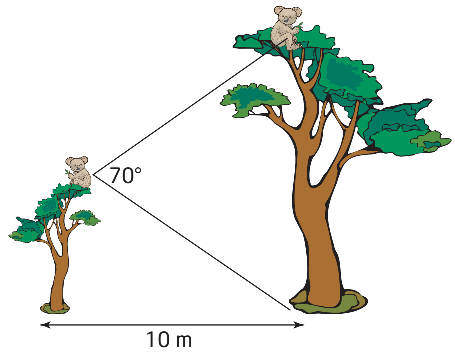


Show how Kirstyn can use the tangent ratio to find the width of the river, to the nearest metre.

10. A surveyor is positioned at a traffic intersection, viewing a marker on the other side of the street. The marker is 18 m from the intersection. The surveyor cannot measure the width directly because there is too much traffic. Find the width of James Street, to the nearest tenth of a metre.



11. Rocco and Biff are two koalas sitting at the top of two eucalyptus trees, which are located 10 m apart, as shown. Rocco's tree is exactly half as tall as Biff's tree. From Rocco's point of view, the angle separating Biff and the base of his tree is 70°.

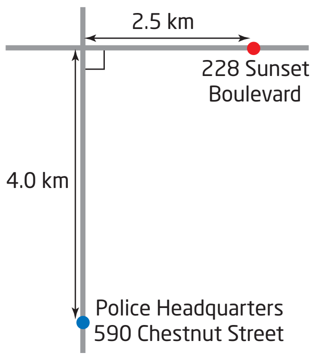


How high off the ground is each koala?

12. Police are responding to a distress call:

Help needed at 228 Sunset
Boulevard immediately.

Police headquarters and the trouble site are shown.



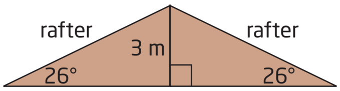
Squad cars and a helicopter are both immediately dispatched to the site from headquarters.

a) At what angle to Chestnut Street should the helicopter travel?

b) Assuming that the squad cars can travel at an average speed of 60 km/h and the helicopter can travel twice as fast, how much longer will it take for the squad cars to reach the site than the helicopter?

c) Describe any assumptions you make in your solutions.

13. The diagram shows the roof of a house. How wide is the house, to the nearest metre?

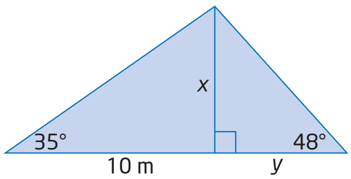


14. Petra walked diagonally across a rectangular schoolyard measuring 45 m by 65 m. To the nearest degree, at what angle with respect to the shorter side did she walk?

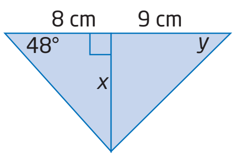
15. Comfortable stairs have a slope of $\frac{3}{4}$.

What angle do the stairs make with the horizontal, to the nearest degree?

16. Find the length of x, then the length of y, to the nearest tenth of a metre.



17. Find the length of x, to the nearest tenth of a centimetre, then the measure of $\angle y$, to the nearest degree.



18. To measure the height of a building, Chico notes that its shadow is 8.5 m long. He also finds that a line joining the top of the building to the tip of the shadow forms a 65° angle with the flat ground.

a) Draw a diagram to illustrate this situation.

b) Find the height of the building, to the nearest tenth of a metre.

19. a) Find the tangent of several angles with values between 1° and 44°. Organize your results in a table like this one.

Angle, θ	tan θ

b) What is the value of tan 45°?

c) Add the tangents of several angles with values between 46° and 89° to your table.

d) Add the tangents of several angles with values very close to, but not equal to, 90° to your table.

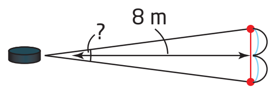
e) Based on your findings, what conclusions can you make about the tangents of angles
- less than 45°?
- greater than 45° and less than 90°?
- very close to, but not equal to, 90°?

f) Use the definition of the tangent ratio and geometric reasoning to justify your conclusions. Include diagrams in your explanation.

20. a) Use a calculator to evaluate the following:
- tan 0°
- tan 90°

b) Use the definition of the tangent ratio and geometric reasoning to explain your results. Include diagrams in your explanation.

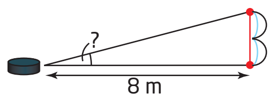
21. At hockey practice, Lars has the puck in front of the net, as shown.



He is exactly 8 m away from the middle of the net, which is 2 m wide. Within what angle must Lars fire his shot in order to get it in the net, to the nearest degree?

22. Refer to question 21.

a) Does Lars have a better chance, a worse chance, or the same chance to score if he positions himself directly in front of one of the posts, as shown? Explain your reasoning and any assumptions you make.



b) Repeat part a) for the case in which Lars moves
- directly closer to the net
- directly farther from the net

Extend

23. a) Make a table of values for tan θ for values of θ between 0° and 90°.

b) Graph the relationship. Is the relationship linear or non-linear? Explain.

c) Describe the shape of the graph and any interesting features you can identify.

24. The angle θ at which a skier slides down a hill with a coefficient of friction, μ, at a constant speed, is given by tan θ = μ. Natalie is skiing on a hill with a reported coefficient of friction of 0.6. If Natalie skis down at a constant speed, what is the angle of the hill?

25. The tangent ratio is used to design the bank angle for a curved section of roadway.

Let θ be the bank angle required for a speed limit, v, in kilometres per hour, and a radius, r, in metres. The angle and the speed limit are related by the formula

$\tan \theta = \dfrac{v^2}{9.8r}$. Find the bank angle required for a highway curve of radius 50 m that will carry traffic moving at 100 km/h.

Did You Know?

This same relation applies to banking a bicycle or motorcycle when going around a curve, and banking an aircraft in a turn.

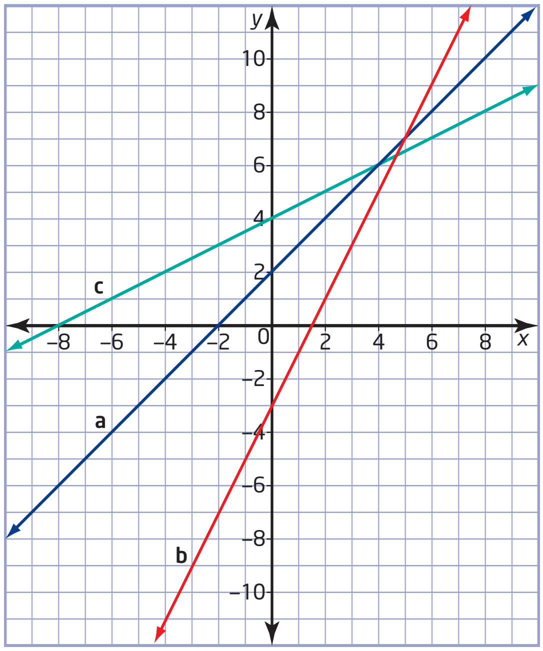
26. For each graph,

i) find the slope of the line

ii) draw a triangle to find the tangent of the acute angle that the line makes with the x-axis

iii) compare your answers to parts i) and ii)

iv) find the acute angle in part ii)



27. Math Contest How many numbers less than 10 000 contain at least one 5?

A 5000

B 6561

C 3439

D 625

E 4944

28. Math Contest In the figure, PR = PQ and ∠RPS = 30°. IF PS = PT, what is the measure of △QST?