7.4 The Sine and Cosine Ratios

We depend on ships and aircraft to transport goods and people all over the world. If you were the captain of a ship or the pilot of an airplane, how could you make sure that you did not get lost in the middle of the ocean? In ancient times, this was a significant problem.

Today, navigational equipment such as Global Positioning System (GPS) devices makes it much easier to find your way around the planet. Even so, factors such as wind and water currents can sometimes complicate travel plans. How can trigonometry help when this happens?

Investigate

Tools

- computer with *The Geometer's Sketchpad®*

OR

- TI-83 Plus or TI-84 Plus graphing calculator

OR

- grid paper
- protractor
- ruler

What are the sine and cosine ratios?

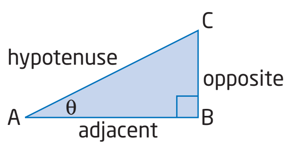
In the last section, you learned the tangent ratio. In this activity, you will investigate two other important ratios. In addition to the opposite and adjacent sides, these ratios involve the third side of a right triangle: the hypotenuse.

1. a) Draw a large right triangle △ABC.

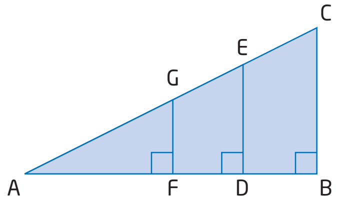
b) Measure the length of the side opposite ∠A.

c) Measure the length of the hypotenuse.

d) Calculate the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$.



2. a) Create overlapping triangles by adding line segments parallel to one of the legs as shown.



b) Explain why these triangles are similar to the first one.

c) Measure and calculate the following for each similar triangle:
- length of the opposite side
- length of the hypotenuse
- the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$

d) Compare the $\frac{\text{opposite}}{\text{hypotenuse}}$ ratios for each triangle and describe what you notice.

3. a) Measure the length of the sides adjacent to $\angle$A for each triangle.

b) Calculate the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ for each triangle. Describe what you notice.

4. a) Draw a new set of similar triangles, with a different value for $\angle$A.

b) Calculate and compare the ratio of $\frac{\text{opposite}}{\text{hypotenuse}}$ for each similar triangle.

c) Calculate and compare the ratio of $\frac{\text{adjacent}}{\text{hypotenuse}}$ for each similar triangle.

d) Repeat for another set of similar triangles.

The two ratios you have just explored are called the sine and cosine ratios. They are defined as

$$\text{sine } A = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{and} \qquad \text{cosine } A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Together with the tangent ratio, they are the three **primary trigonometric ratios**.

5. Reflect Summarize what you have discovered about the three primary trigonometric ratios, using words and diagrams.

Technology Tip

To do step 2 using dynamic geometry software, follow these steps:
- Select the horizontal line segment. Construct a point on it.
- Select this segment and the new point, and construct a perpendicular line.
- Select the perpendicular line and the hypotenuse. Construct the point of intersection.
- Hide the perpendicular line. Construct a line segment connecting the two new points.

Literacy Connections

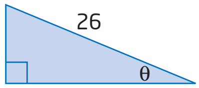
The short forms for sine and cosine are sin and cos.

primary trigonometric ratios
- sine, cosine, and tangent
- often abbreviated as sin, cos, and tan

Example 1 Find the Primary Trigonometric Ratios

Find the three primary trigonometric ratios for θ. Express the ratios as decimals, rounded to four decimal places.

a)



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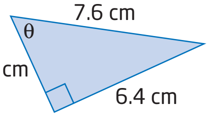
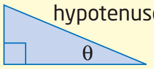


Solution

I need to identify the opposite, adjacent, and hypotenuse sides relative to the angle θ.



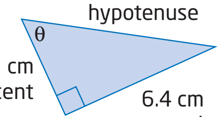
a)

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{10}{26} = \frac{5}{13} \doteq 0.3846$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{24}{26} = \frac{12}{13} \doteq 0.9231$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{10}{24} = \frac{5}{12} \doteq 0.4167$$

b)



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{6.4}{7.6} \doteq 0.8421$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4.1}{7.6} \doteq 0.5395$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{6.4}{4.1} \doteq 1.5610$$

Since sine, cosine, and tangent are ratios, they have no units.

Literacy Connections

A memory device (or mnemonic) for the three primary trigonometric ratios uses these short forms:

$S = \frac{O}{H}$ $C = \frac{A}{H}$ $T = \frac{O}{A}$

These short forms produce the nonsense phrase soh cah toa. This phrase may help you remember the formulas for the trigonometric ratios.

Making Connections

Trigonometric ratios are often expressed with three or four digits of accuracy. This is to ensure that angles are found with enough precision. Consider the difference in the following two calculations:

$\tan^{-1}(0.2) \doteq 11.310°$ $\tan^{-1}(0.3) \doteq 16.699°$

Compare the difference between these two angles:

$\tan^{-1}(0.2492) \doteq 13.993°$ $\tan^{-1}(0.2493) \doteq 13.998°$

In both cases, the tangents of the angles differ by one decimal place. But in the second case, the angles are much closer together. Possible discrepancies due to rounding are reduced by carrying more digits until the final step in a calculation.

Just as with the tangent ratio, you can find the sine and cosine of an angle using a scientific or graphing calculator.

Example 2 Find the Sine and Cosine of an Angle

Evaluate the following, to four decimal places.

a) $\sin 26°$ **b)** $\cos 75°$

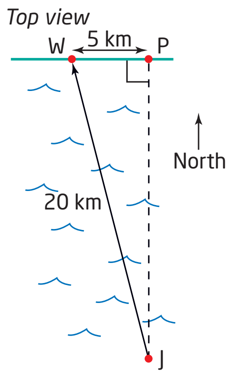
Solution

Make sure that your calculator is in degree mode.

a) $\sin 26° \doteq 0.4384$ **b)** $\cos 75° \doteq 0.2588$

Example 3 Find an Angle Using the Sine and Cosine Ratios

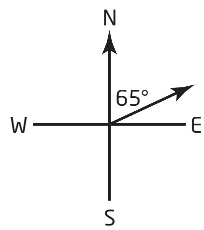
a) Captain Jack is navigating his ship to Port Harbour, which is directly north of the ship's location. To compensate for an easterly current, he aims for a point on shore that is 5 km west of Port Harbour. Assuming that the point on shore is 20 km from his position now, at what bearing must Jack head his ship?



b) Captain Jack is in communication with a submarine that is cruising at a depth of 400 m below sea level. If Jack's radar tells him that the submarine is 500 m from Jack, due north of his ship, at what angle is the submarine located with respect to Captain Jack's ship, to the nearest degree?

Literacy Connections

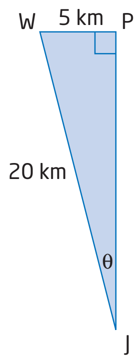
Bearing is a navigational term that describes a direction. It is expressed as an angle in terms of north, south, east, and/or west.



This bearing can be described as N65°E, which is read as "65 degrees east of north."

Solution

a) Captain Jack's ship, Port Harbour, and the western target form a right triangle.



For the unknown bearing angle, the opposite and hypotenuse sides are known.

Apply the sine ratio to find Jack's bearing.

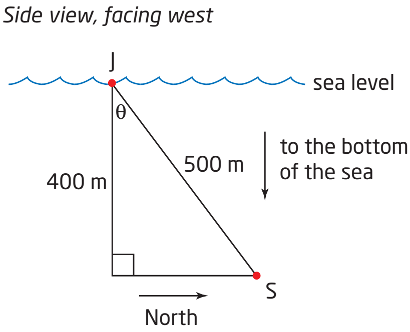
$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$= \frac{5}{20}$$
$$= 0.25$$

To find θ, calculate the inverse sine of 0.25.

$\theta = \sin^{-1}(0.25)$
$\doteq 14.477°$

Jack must head his ship on a bearing of approximately N15°W.

b) Draw a diagram that shows the relative positions of Captain Jack's ship and the submarine. Captain Jack's ship, the submarine, and a line segment that points straight down from Captain Jack's ship form a right triangle.



For the unknown angle relating the submarine's position to Captain Jack, the adjacent and hypotenuse sides are known. Apply the cosine ratio to find this angle.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{400}{500}$$

$$\theta = \cos^{-1}\left(\frac{400}{500}\right)$$

$$\doteq 36.870$$

2nd [COS⁻¹] (400 ÷ 500) =

or (400 ÷ 500) 2nd [COS⁻¹]

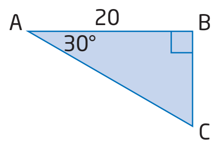
The submarine is approximately 37° north of Captain Jack's ship with respect to a line that points straight down to the bottom of the sea.

Example 4 Solve a Right Triangle

Literacy Connections

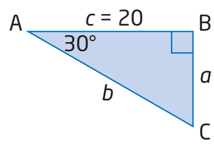
A triangle has six measures: three side lengths and three angle measures. To solve a triangle means to find all six of these values.

Solve △ABC. Round side lengths to the nearest unit and angles to the nearest degree.



Solution

Label the sides according to their corresponding angles.



Use the two known angles to find ∠C.

$$\angle A + \angle B + \angle C = 180°$$
$$30° + 90° + \angle C = 180°$$
$$120° + \angle C = 180°$$
$$\angle C = 180° - 120°$$
$$\angle C = 60°$$

Note that in a right triangle, the two acute angles are complementary. You can find ∠C by subtracting: 90° − 30° = 60°.

Literacy Connections

When naming parts of triangles, use capital letters to represent the angles and vertices and corresponding lowercase letters to represent opposite sides.

Use the cosine ratio to find side b.

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos A = \frac{c}{b}$$

$$\cos 30° = \frac{20}{b}$$

$$b(\cos 30°) = 20$$ **Multiply both sides by *b*.**

$$b = \frac{20}{\cos 30°}$$ **Divide both sides by cos 30°.**

$$b \doteq 23.094$$ 20 [÷] [COS] 30 [=] or 20 [÷] 30 [COS] [=]

Use the tangent ratio to find a.

$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$

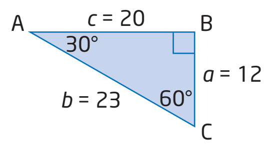
$$\tan A = \frac{a}{c}$$

$$\tan 30° = \frac{a}{20}$$

$$20(\tan 30°) = a$$ **Multiply both sides by 20.**

$$a \doteq 11.547$$ 20 [×] [TAN] 30 [=] or 20 [×] 30 [TAN] [=]

The diagram shows the solved triangle.



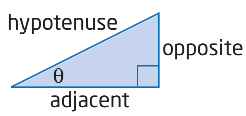
There is often more than one way to solve a right triangle. If you know the value of any side plus an additional side or angle, you can find the other measures. Notice, however, that some answers may be slightly different, due to rounding in the intermediate steps of a solution.

Key Concepts

- The three primary trigonometric ratios are sine, cosine, and tangent. They are defined as follows:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

- You can find any side length or angle measure of a right triangle if you know two pieces of information in addition to the right angle.



Communicate Your Understanding

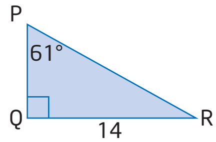
C1 Explain why the primary trigonometric ratios depend only on a given angle and not the size of a right triangle.

C2 **a)** Create a problem for which you would need to apply the cosine function on your calculator. Solve the problem and explain each step.

b) Repeat part a) for the inverse sine function of your calculator.

C3 **a)** List the steps you would use to solve △PQR.

b) List a different set of steps to solve △PQR using another method.

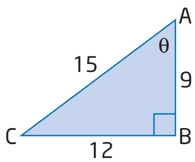


Practise

For help with questions 1 and 2, see Example 1.

1. Find sin θ, cos θ, and tan θ for each triangle, expressed as fractions in lowest terms.

a)



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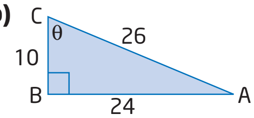

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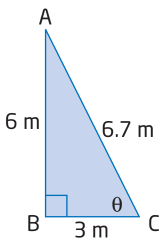

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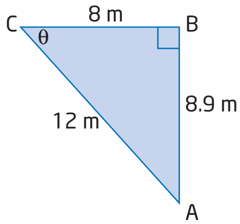

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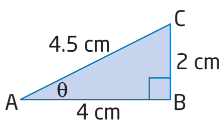

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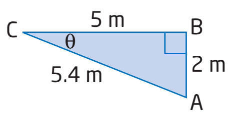

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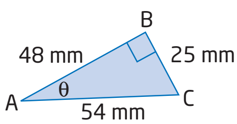

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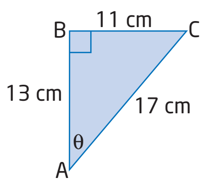
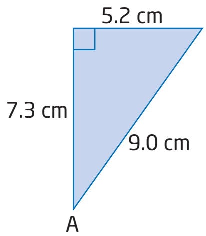


2. Find the three primary trigonometric ratios for ∠A, to four decimal places.

a)



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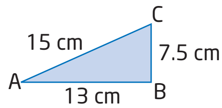

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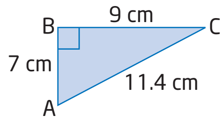

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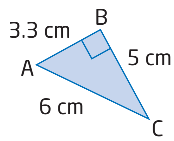

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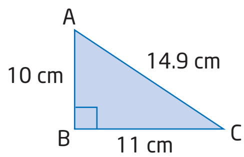

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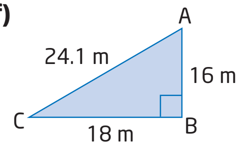


For help with questions 3 to 5, see Example 2.

3. Evaluate each of the following with a calculator, rounded to four decimal places.

a) sin 35°
b) sin 45°
c) sin 60°
d) sin 37°
e) sin 25°
f) sin 0°
g) sin 89°
h) sin 30°

4. Evaluate each of the following with a calculator, rounded to four decimal places.

a) $\cos 80.2°$ b) $\cos 45°$
c) $\cos 30°$ d) $\cos 60°$
e) $\cos 89°$ f) $\cos 0°$
g) $\cos 5°$ h) $\cos 83°$

5. Compare your results to questions 3 h) and 4 d). Use a diagram to help explain these results.

For help with questions 6 to 9, see Example 3.

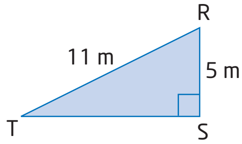
6. Find the measure of each angle, to the nearest degree.

a) $\sin \theta = 0.8933$ b) $\sin \theta = 0.5032$
c) $\sin P = \frac{1}{2}$ d) $\sin S = \frac{2}{3}$
e) $\sin \theta = \frac{3}{4}$ f) $\sin A = 0.9511$
g) $\sin \theta = 0.7123$ h) $\sin \theta = \frac{2}{5}$
i) $\sin X = 0.3035$ j) $\sin \theta = 0.9976$
k) $\sin V = \frac{1}{8}$ l) $\sin \theta = 0$

7. Find the measure of each angle, to the nearest degree.

a) $\cos \theta = 0.4481$ b) $\cos A = 0.6329$
c) $\cos C = \frac{5}{11}$ d) $\cos \theta = 0.3432$
e) $\cos Q = 0.8871$ f) $\cos M = \frac{3}{14}$
g) $\cos \theta = \frac{1}{6}$ h) $\cos \theta = 0.6215$
i) $\cos B = \frac{15}{16}$ j) $\cos X = 0.0193$
k) $\cos \theta = 0$ l) $\cos J = \frac{1}{2}$

8. Calculate sin T in each triangle. Then, find ∠T, to the nearest degree.

a)


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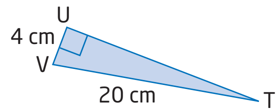
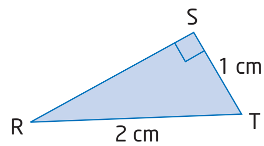


9. Calculate cos T in each triangle. Then, find ∠T, to the nearest degree.

a)


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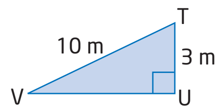
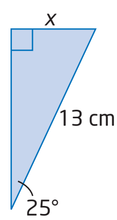


For help with questions 10 to 14, see Example 4.

10. Find the length of x, to the nearest tenth of a unit, by applying the sine ratio.

a)


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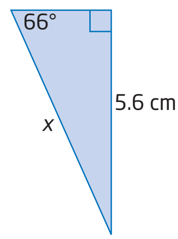
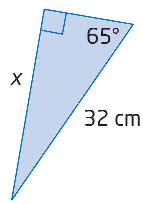


c)

d)


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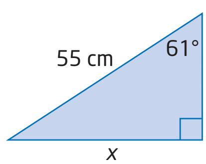

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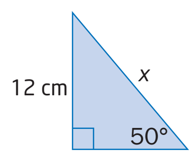

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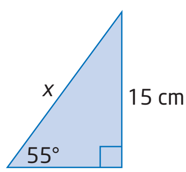

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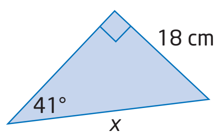
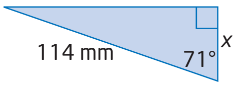
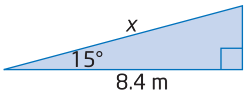


11. Find the length of x, to the nearest tenth of a unit, by applying the cosine ratio.

a)



b)



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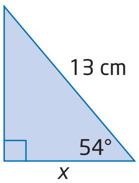

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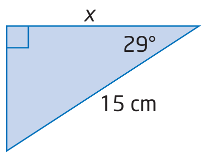

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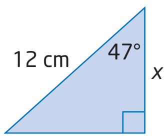

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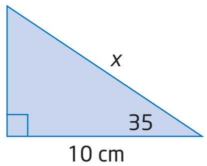

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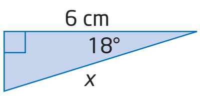

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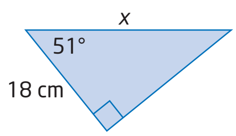
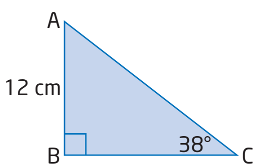


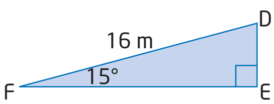
In questions 12 to 14, round side lengths to the nearest tenth of a unit and angles to the nearest degree.

12. Solve each triangle.

a)



b)



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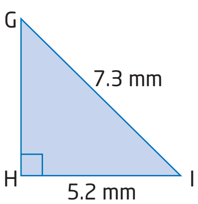
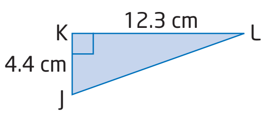


d)



13. In △DEF,
DF = 6.0 km
∠E = 44°
∠F = 90°

a) Draw this triangle and label the given information.

b) Solve △DEF.

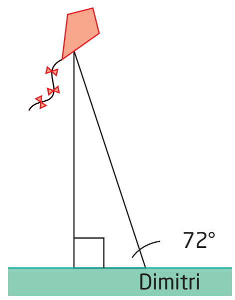
14. In △XYZ,
XY = 16 cm
YZ = 11 cm
∠Z = 90°

a) Draw this triangle and label the given information.

b) Solve △XYZ.

Connect and Apply

15. Dmitri has let out 40 m of his kite string, which makes an angle of 72° with the horizontal ground.



a) Find the height of the kite, to the nearest metre.

b) Suppose the Sun is shining directly above the kite. How far is the kite's shadow from Dmitri, to the nearest metre?

16. During take-off a plane must rise at least 20 m during its first 1.5 km of flight to successfully clear the runway.

a) At what minimum average angle must the plane climb for a safe take-off, to the nearest hundredth of a degree?

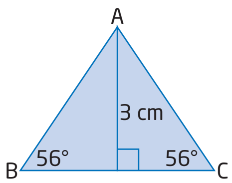
b) If the required rise is doubled to 40 m, does this double the climb angle? Explain.

17. In $\triangle PQR$, $\angle Q = 90°$ and $PR = 20$ cm. Find PQ, to the nearest tenth of a centimetre, if $\angle R = 41°$.

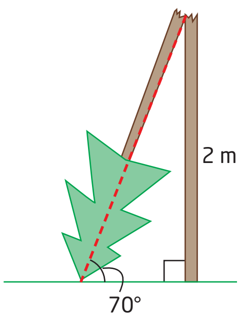
18. In $\triangle DEF$, find $\angle F$, to the nearest degree, if $DE = 15$ cm, $DF = 18$ cm, and $\angle E = 90°$.

19. The towrope pulling a parasailor is 70 m long. A boat crew member estimates that the angle between the towrope and the water is about 30°. Find the height of the parasailor above the water.

20. $\triangle ABC$ is an isosceles triangle. The height of the triangle is 3 cm, and the two acute angles at its base are each 56°. How long are the two equal sides, to the nearest tenth of a centimetre?



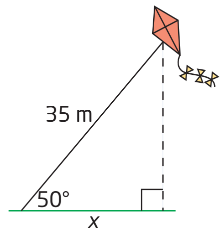
21. A tree is splintered by lightning 2 m up its trunk, so that the top part of the tree touches the ground. The angle the top of the tree forms with the ground is 70°. Before it was splintered, how tall was the tree, to the nearest tenth of a metre?



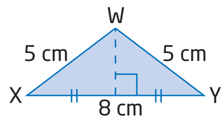
22. The side adjacent to the 74° angle in a right triangle is 6 cm long. How long is the hypotenuse, to the nearest tenth of a centimetre?

23. The hypotenuse of a right triangle is 10 m long. How long is the side adjacent to the 21° angle, to the nearest tenth of a metre?

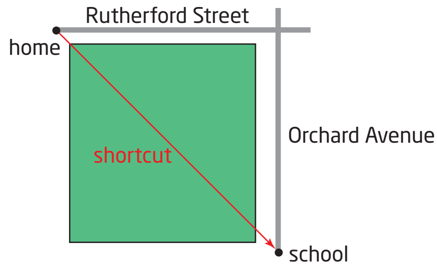
24. A kite string is 35 m long. The angle the string makes with the ground is 50°. To the nearest metre, how far from the person holding the string is a person standing directly under the kite?



25. Find all the angles in $\triangle WXY$, to the nearest degree.



26. To get to school, Enzo can travel 1.2 km east on Rutherford Street and then south on Orchard Avenue to his school. Or, he can take a shortcut through the park, as shown. His shortcut takes him 20 min.

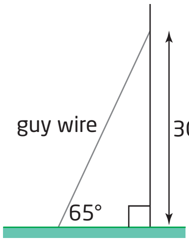


Enzo's walking speed is 6 km/h.

a) What angle does Enzo's shortcut make with Rutherford Street? Describe any assumptions you must make.

b) How much time does Enzo save by taking his shortcut? Explain your answer.

27. In Example 5, Section 7.3, this problem was solved. A radio transmitter is to be supported with a guy wire. The wire is to form a 65° angle with the ground and reach 30 m up the transmitter. The wire can be ordered in whole-number lengths of metres. How much wire should be ordered?



Solve this problem using a more efficient method.

28. How are the sines and cosines of the acute angles of a right triangle related? Plan and carry out an investigation to explore this. Write a brief report of your findings, using words, diagrams, and mathematical notation.

29. **a)** Is it possible for the sine or cosine of an angle to be greater than 1? Use geometric reasoning to explain your answer.

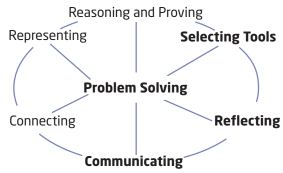
b) Is it possible for the tangent of an angle to be greater than 1? Use geometric reasoning to explain your answer.

30. **Chapter Problem** A right triangle is formed by the following locations:
- your current location (in the race)
- the capital of the United States
- your next destination

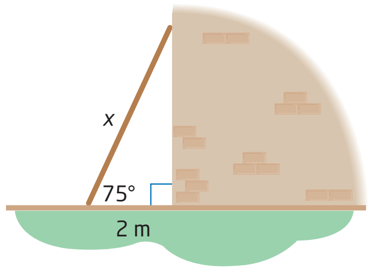
The cosine of the angle at your current location is approximately 0.8. What city is your next destination, and how far do you have to travel to get there?

Achievement Check

31. When it is leaning against a wall, the foot of a ladder is 2 m from the base of the wall. The angle between the ladder and the ground is 75°.



a) How high up the wall does the ladder reach, to the nearest centimetre?

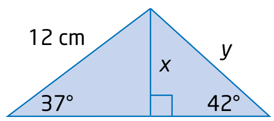


b) How long is the ladder, to the nearest centimetre?

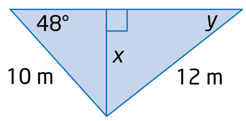
c) If the ladder slips down the wall so that it makes an angle of 55° with the ground, does the end on the ground slip more than the end against the wall? Explain.

Extend

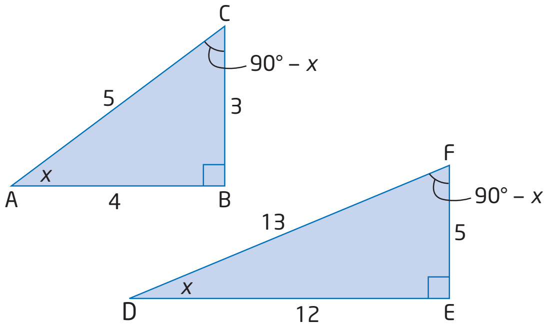
32. Find the length of x, then the length of y, to the nearest tenth of a centimetre.



33. Find the length of x, to the nearest tenth of a metre, then the measure of y, to the nearest degree.



34. a) Use △ABC and △DEF. Copy and complete the table. Leave all ratios in fraction form.



Triangle	△ABC	△DEF
tan x		
sin x		
cos x		
tan $(90° - x)$		
sin $(90° - x)$		
cos $(90° - x)$		

b) How is tan x related to tan $(90° - x)$ in these two triangles?

c) How is sin x related to cos $(90° - x)$ in these two triangles?

d) How is cos x related to sin $(90° - x)$ in these two triangles?

e) Explain the relationships in parts b), c), and d).

35. Use geometric reasoning to show that $\sin\theta = \cos(90° - \theta)$ in all right triangles, if θ is one of the acute angles.

36. Use algebraic reasoning to show that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

37. a) Select three different values for an angle x between 0° and 90°. For each value of x, evaluate the expression $(\sin x)^2 + (\cos x)^2$.

b) Use your results from part a) to write a conjecture.

c) Verify your conjecture from part b) using geometric reasoning.

38. Pilots use a "wind triangle" to determine which way to aim the aircraft to overcome the effects of wind. For example, Seymour has an airplane that cruises at 200 km/h in still air. A stiff wind of 28 km/h is blowing from the west. Seymour wants to fly from A directly north to B.

a) Find the angle at which Seymour must aim the airplane.

b) How fast will he be flying relative to the ground?

39. Math Contest At the senior prom, four couples are seated randomly around a circular table. What is the probability that Dan sits beside his date Ranjit?

A $\frac{1}{4}$

B $\frac{1}{8}$

C $\frac{1}{2}$

D $\frac{2}{7}$

E $\frac{3}{8}$

40. Math Contest An equilateral triangle and a hexagon have equal perimeters. The area of the triangle is 2 m^2. What is the area of the hexagon?

41. Math Contest The lengths of the sides of a triangle are 20 cm, 21 cm, and 29 cm. The shortest distance from the longest side to the opposite vertex is

A $\frac{400}{9}$ cm

B $\frac{410}{29}$ cm

C $\frac{420}{29}$ cm

D $\frac{580}{21}$ cm

E $\frac{609}{20}$ cm