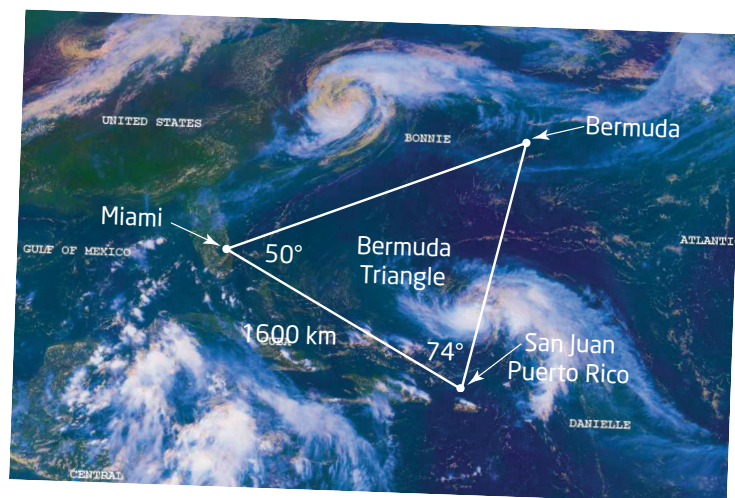# The Sine Law

The Bermuda Triangle, in the north Atlantic Ocean, is the location of several unexplained plane and ship disappearances. Various theories have been suggested to explain this mysterious phenomenon, such as volatile weather patterns, magnetic compass variation, human error, and even alien abduction.

Suppose you took a cruise around the Bermuda Triangle. How could you determine its perimeter from the given information?

■ Investigate

Tools
- protractor and ruler

OR

- computer with *The Geometer's Sketchpad®*

Technology Tip

If you are using *The Geometer's Sketchpad®*, you can access the sine function from the **Functions** drop-down menu on the calculator.

Technology Tip

Instead of constructing new triangles to compare results, simply move one of the vertices of △ABC to examine other acute triangles. The calculated measures will change automatically.

How are the side lengths and sines of angles related in an acute triangle?

1. **a)** Draw an acute △ABC.

   b) Measure the side lengths and the angles.

   c) Calculate the sine of each angle.
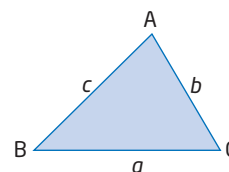
2. **a)** Calculate the ratios $\dfrac{a}{b}$, $\dfrac{\sin A}{\sin B}$, $\dfrac{a}{c}$, $\dfrac{\sin A}{\sin C}$, $\dfrac{b}{c}$, and $\dfrac{\sin B}{\sin C}$.

   b) Do you notice any relationships between the ratios? Explain.

3. **a)** Calculate each ratio.

   $$\frac{a}{\sin A} \qquad \frac{b}{\sin B} \qquad \frac{c}{\sin C}$$

   b) Compare these results and explain what you notice.

4. Repeat steps 1 to 3 for two different triangles. Are the results the same? Explain.

5. **Reflect** Summarize the relationship of the sides and sines of angles in an acute triangle.

The relationship you investigated is called the **sine law** . To show why the sine law holds true, draw an acute triangle and add an altitude, h, from one of the vertices.

The altitude splits $\triangle ABC$ into two smaller right triangles, $\triangle AXC$ and $\triangle BXC$.

Find an expression for h.

sine law

- the relationship between the sides and their opposite angles in any acute $\triangle ABC$:
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Focus on $\triangle AXC$:

$$\sin A = \frac{h}{b} \qquad \textbf{Multiply both sides by } \textbf{\textit{b}}.$$

$$b(\sin A) = h$$

Focus on $\triangle BXC$:

$$\sin B = \frac{h}{a} \qquad \textbf{Multiply both sides by } \textbf{\textit{a}}.$$

$$a(\sin B) = h$$

Set the two expressions for h equal.

$$b(\sin A) = a(\sin B)$$

$$\frac{b(\cancel{\sin A})}{\cancel{\sin A}} = \frac{a(\sin B)}{\sin A} \qquad \textbf{Divide both sides by sin A.}$$

$$b = \frac{a(\sin B)}{\sin A}$$

$$\frac{b}{\sin B} = \frac{a(\cancel{\sin B})}{\sin A(\cancel{\sin B})} \qquad \textbf{Divide both sides by sin B.}$$

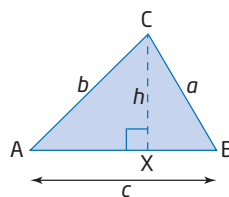$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

This process can be repeated using a different altitude.

Combining the results gives the sine law.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Even though there are three parts to this equation, you only use two parts at a time. The choice of which two to use depends on what information is given.
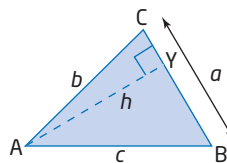
You can apply the sine law to find an unknown side length of an acute triangle if you know two angles and one of the side lengths.
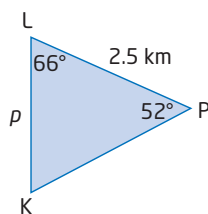
Example 1 Find a Side Length Using the Sine Law

A bicycle path forms a 66° angle with one lock of a canal. At a distance of 2.5 km along the bicycle path, the angle separating this lock from the next lock is 52°. How far apart are the two locks, to the nearest tenth of a kilometre?
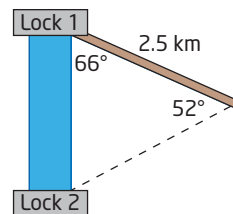
Solution

Draw a simplified diagram showing the given information.

Before you can use the sine law to find length p, you need to find the measure of $\angle K$.

$\angle K = 180° - 66° - 52°$ **The sum of the interior angles in a triangle is 180°.**
$\quad\ = 62°$

Now, apply the sine law.

$$\frac{p}{\sin P} = \frac{k}{\sin K}$$

$$\frac{p}{\sin 52°} = \frac{2.5}{\sin 62°}$$ **Substitute the given information.**

$$\cancel{\sin 52°}\left(\frac{p}{\cancel{\sin 52°}}\right) = \sin 52°\left(\frac{2.5}{\sin 62°}\right)$$ **Multiply both sides by sin 52°.**

$$p = \sin 52°\left(\frac{2.5}{\sin 62°}\right)$$

$$p \doteq 2.2$$

```
sin(52)*2.5/sin(
62)
        2.231193448
```

The distance between the two locks is 2.2 km, to the nearest tenth of a kilometre.

You can also use the sine law to find an unknown angle if you know two sides and an angle opposite one of the known sides.

Sometimes the sine law appears in a different form. Recall the result obtained earlier for △ABC.
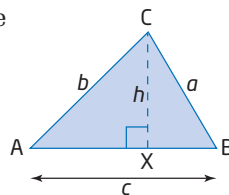
$b(\sin A) = a(\sin B)$

$$\frac{b(\sin A)}{ab} = \frac{a(\sin B)}{ab}$$ **Divide both sides by _ab_.**

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Combining the results gives an alternative form of the sine law.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Either form can be used to find a missing side or angle—this alternative form is a little easier to use when finding an unknown angle.

Example 2 Find an Angle Using the Sine Law

In acute △TUV, TU = 11.1 cm, UV = 5.8 cm, and ∠T = 31°. Find the measure of ∠V, to the nearest degree.

Solution

Draw a diagram and label the given information.
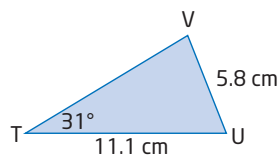
Apply the sine law.

$$\frac{\sin V}{v} = \frac{\sin T}{t}$$

$$\frac{\sin V}{11.1} = \frac{\sin 31°}{5.8}$$ **Substitute the given information.**

$$11.1\left(\frac{\sin V}{11.1}\right) = 11.1\left(\frac{\sin 31°}{5.8}\right)$$ **Multiply both sides by 11.1.**

$$\sin V = 11.1\left(\frac{\sin 31°}{5.8}\right)$$

$\sin V = 0.9856\ldots$

$\angle V = \sin^{-1}(0.9856\ldots)$ **Use the inverse sine operation.**

$\angle V \doteq 80°$

```
11.1*sin(31)/5.8
          .9856763158
sin-1(Ans)
          80.29077071
```
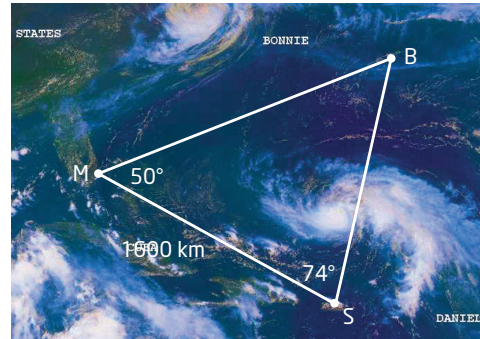
The measure of ∠V is about 80°.

Example 3 Perimeter of the Bermuda Triangle

Use the information given on the diagram to determine the perimeter of the Bermuda Triangle, to the nearest hundred kilometres.
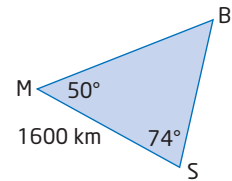

Solution

Draw a simplified diagram showing the given information.

First, find $\angle B$.

$\angle B = 180° - 50° - 74°$
$\quad = 56°$

Now, apply the sine law twice to find sides m and s.

$\dfrac{m}{\sin M} = \dfrac{b}{\sin B}$

$\dfrac{m}{\sin 50°} = \dfrac{1600}{\sin 56°}$

$\qquad m = \sin 50° \left(\dfrac{1600}{\sin 56°} \right)$

$\qquad m \doteq 1478$

Method 1: Use Side b

$\dfrac{s}{\sin S} = \dfrac{b}{\sin B}$

$\dfrac{s}{\sin 74°} = \dfrac{1600}{\sin 56°}$

$\qquad s = \sin 74° \left(\dfrac{1600}{\sin 56°} \right)$

$\qquad s \doteq 1855$

Method 2: Use Side m

$\dfrac{s}{\sin S} = \dfrac{m}{\sin M}$

$\dfrac{s}{\sin 74°} = \dfrac{1478}{\sin 50°}$

$\qquad s = \sin 74° \left(\dfrac{1478}{\sin 50°} \right)$

$\qquad s \doteq 1855$

To find the perimeter, add the sides.

$P = b + m + s$
$\quad = 1600 + 1478 + 1855$
$\quad = 4933$

Therefore, the perimeter of the Bermuda Triangle is approximately 4900 km.

Key Concepts

- In an acute $\triangle$ABC, the sine law states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
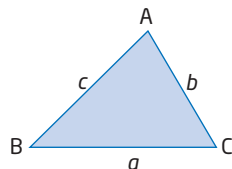
- The sine law can be used to find
  - an unknown side if two angles and a side are known
  - an unknown angle if two sides and the angle opposite one of the known sides are known

- The sine law can also be written in the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Communicate Your Understanding

C1 **a)** Why does the sine law have three parts to its equation?

b) Do you use all three parts at once? How can you tell which parts to use?

C2 Which of the following triangles can be solved using the sine law? For those that can, explain how. For those that cannot, explain why not.
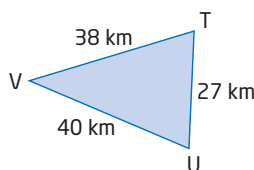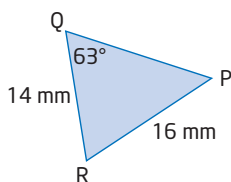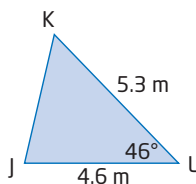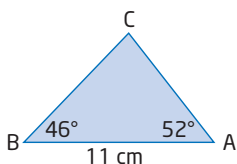
C3 How many pieces of information about a triangle's side lengths and angles are needed in order to solve it using the sine law? What possible combinations will work?

C4 There are two forms of the sine law, relating sides a and b and $\angle$A and $\angle$B:

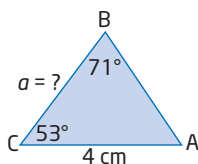$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{and} \quad \frac{\sin A}{a} = \frac{\sin B}{b}$$

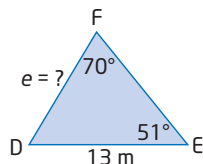Explain when it is easier to apply each form. Create examples to support your explanation.

For help with questions 1 and 2, see Example 1.

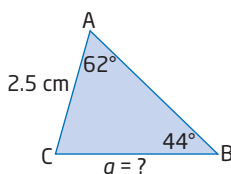**1.** Find the length of the indicated side in each triangle, to the nearest unit.
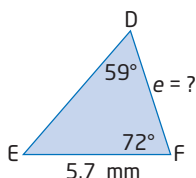
a)

b)

2. Find the length of the indicated side in each triangle, to the nearest tenth of a unit.
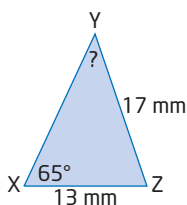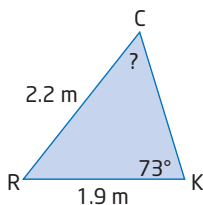
a)

b)

For help with questions 3 and 4, see Example 2.

3. Find the measure of the indicated angle in each triangle, to the nearest degree.

a)

b)

4. Draw a diagram and label the given information. Then, find the measure of the indicated angle in each triangle, to the nearest degree.
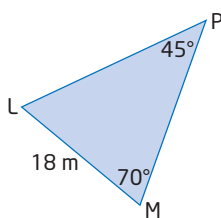
a) In acute $\triangle GHK$, $\angle G = 47°$, $h = 5$ cm, and $g = 4$ cm. Find $\angle H$.

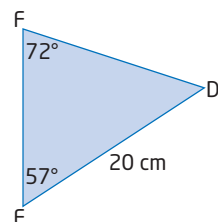**b)** In acute $\triangle RST$, $\angle S = 72°$, $t = 1.5$ m, and $s = 1.8$ m. Find $\angle T$.

For help with questions 5 to 7, see Example 3.

5. Solve each triangle. Round answers to the nearest unit, if necessary.
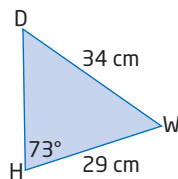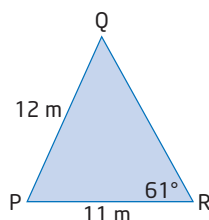
a)

b)

6. Solve each triangle. Round answers to the nearest unit, if necessary.

a)

b)

7. Draw a diagram and label the given information. Then, solve each triangle. Round answers to the nearest unit, if necessary.

a) In acute $\triangle AKR$, $k = 15$ mm, $r = 13$ mm, and $\angle K = 68°$.

b) In acute $\triangle UJF$, $j = 23$ km, $\angle U = 57°$, and $\angle F = 48°$.

8. **Use Technology** Check your answers to question 7 using dynamic geometry software.

Connect and Apply

9. A sign is supported by a pole and a cable, as shown. The cable is attached to the wall 2.2 m above the base of the pole.
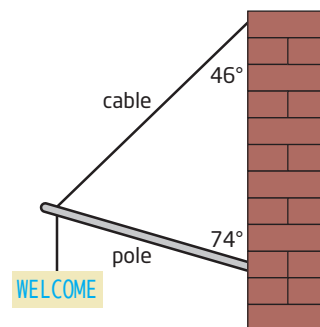
a) Find the length of the pole, to the nearest tenth of a metre.

b) Find the length of the cable, to the nearest tenth of a metre.

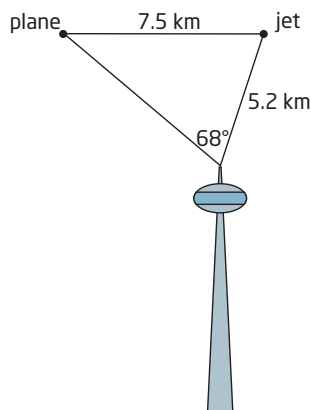**10.** A small commercial plane and a jet airliner are 7.5 km from each other, at the same altitude. From an observation tower, the two aircraft are separated by an angle of 68°. If the jet airliner is 5.2 km from the observation tower, how far is the commercial plane from the observation tower, to the nearest tenth of a kilometre?


11. A telephone pole is supported by an 18-m guy wire that makes an angle of 50° with the horizontal ground. A 14-m guy wire is to be fastened on the other side of the pole for reinforcement. Both wires attach to the pole at its top. Round your answers to the nearest unit, if necessary.

a) What angle should the second wire make with the ground?

b) How tall is the pole?

c) How far is the base of each wire from the base of the pole?

**d)** Could you solve this problem without using the sine law? Explain.

12. A bridge across a valley is 150 m in length. The valley walls make angles of 60° and 54° with the bridge that spans it, as shown. How deep is the valley, to the nearest metre?
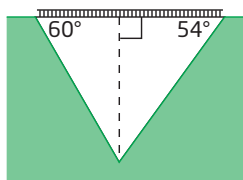
13. Chapter Problem You and your partner are observing an aircraft from two observation decks, located 5.0 km apart. From your point of view, the aircraft is at an angle of elevation of 70°. From your partner's point of view, the angle of elevation is 55°. Determine the altitude of the aircraft, to the nearest tenth of a kilometre.
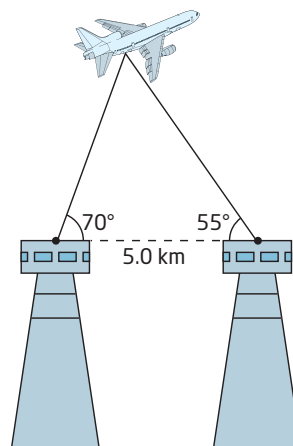
14. In isosceles △ABC, $c = 15$ cm, $a = 11$ cm, and ∠B = 68.5°. Can this triangle be solved using the sine law? If so, solve it and explain your reasoning. If not, explain why not.

15. To measure the height of a water tower, Kelly walks 0.15 km from the base of the tower along an inclined path to point P. From P, the top of the tower appears at an angle of elevation of 12°.
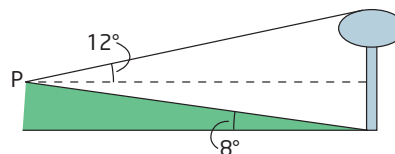
Find the height of the water tower, to the nearest metre.

16. You can find the area of a triangle if you know all three of its side lengths by applying Heron's formula,

$A = \sqrt{s(s - a)(s - b)(s - c)}$, where a, b, and c are the side lengths, and

$s = \dfrac{1}{2}(a + b + c)$.

a) Use Heron's formula to find the area of the Bermuda Triangle.

b) Check this result by estimating the height of the triangle and applying the formula for the area of a triangle,

$A = \dfrac{bh}{2}$.

> **Did You Know?**
>
> Heron was a mathematician, engineer, and inventor born in Alexandria, Greece, during the first century. Among his accomplishments was the invention of the first steam engine, called an *aelopile*. The principle Heron used in the aelopile is similar to that used in today's jet engines.

17. Refer to question 16. Lan's garden is in the shape of an acute triangle. Two of the vertices have angles of 55° and 58°. The side joining these vertices is 6.2 m in length. What is the area of this garden, to the nearest tenth of a square metre?

18. a) Create a problem involving the sine law for which the answer is 15 m.

b) Solve the problem.

c) Trade with a partner and solve each other's problem. Check your solutions.

19. Does the sine law work if you replace sines with cosines or tangents? Investigate several triangles and explain whether there is such a relationship.

Extend

20. a) Show that the sine law simplifies to the sine ratio when one of the angles of a triangle is 90°.

b) Would you use the sine law to solve right triangles? Explain.

21. Refer to question 16. Show that Heron's formula can also be written as follows:

$A = \dfrac{1}{4}\sqrt{(a + b + c)(a + b - c)(b + c - a)(c + a - b)}$

22. Math Contest Frishta, Michelle, Daniel, and Vahpav each shoot two arrows at the target shown. Each ring has a different point value.
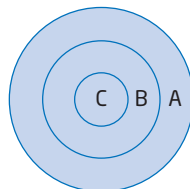
- Frishta shoots one arrow in ring B and one in ring C. Her score is 17.
- Michelle shoots one in ring A and one in ring B. Her score is 12.
- Daniel shoots one in ring C and one in ring A. His score is 15.
- Vahpav shoots two arrows into ring A.

Vahpav's score is

A 8

B 10

C 12

D 18

E 20

23. Math Contest When each of three different numbers is added to the average of the other two, the results are 105, 106, and 125. What is the average of the three original numbers?

A 51

B 55

C 56

D 57

E 63