9.3 Minimize the Surface Area of a Square-Based Prism

The boxes used in packaging come in many shapes and sizes. A package must be suitable for the product, visually appealing, and cost efficient. Many manufacturers and consumers are conscious of our environment and want to conserve materials whenever possible.

Investigate

How can you compare the surface areas of square-based prisms with the same volume?

Tools
- interlocking cubes

Method 1: Use Manipulatives

1. Use 16 interlocking cubes to build as many different square-based prisms as possible with a volume of 16 cubic units. Calculate the surface area of each prism. Record your results in a table.

Length	Width	Height	Volume	Surface Area
1	1	16	16	

2. Is it possible to build a prism with this volume that has a square base with three cubes on each side? Explain.

minimum
- least possible

3. Which square-based prism has the **minimum**, or optimal, surface area? Describe the shape of this prism compared to the others.

4. **a)** Predict the dimensions of the square-based prism with minimum surface area if you use 64 cubes.
   b) Test your prediction by completing a similar table.

5. Predict the dimensions of the square-based prism with minimum surface area if you use 27 cubes. What about 125 cubes?

6. **Reflect** Summarize your findings. Describe any relationship you notice between the length, width, and height of a square-based prism with minimum surface area for a given volume.

Tools

- spreadsheet software
- computers

Method 2: Use a Spreadsheet

1. Set up a spreadsheet to automatically calculate the height and surface area of a square-based prism given the dimensions of the square base and the volume.

	A	B	C	D	E
1	Side Length of Square Base (cm)	Area of Square Base (cm^2)	Height (cm)	Volume (cm^3)	Surface Area (cm^2)
2	1			64	
3	2			64	
4					

2. Enter the formula for the area of the square base.

3. The volume of the prism is always 64 cm^3. Explain why the height can be found using the expression 64/(area of the square base). Enter this expression as a formula in the Height column.

4. The surface area of a square-based prism consists of the two square ends and the four rectangular sides. The surface area can be found using the expression 2*(column B) + 4*(column A)*(column C). Explain why. Enter this expression as a formula in the Surface Area column.

5. Use the spreadsheet to create different square-based prisms with a volume of 64 cm^3. Begin with a prism that has a square base with sides of 1 cm. Next, increase the dimensions of the square base to 2 cm, then 3 cm, and so on. Use **Fill Down** to complete the spreadsheet.

6. Which prism has the least surface area? Describe the shape of this prism.

7. **a)** Predict the dimensions of the square-based prism with minimum surface area if the volume of the prism is 125 cm^3.
   b) Test your prediction using a spreadsheet.

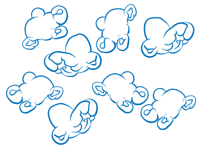
8. **a)** Predict the dimensions of the square-based prism with minimum surface area if the volume is 300 cm^3.
   b) Test your prediction using the spreadsheet. Enter dimensions for the base that are not whole numbers, trying to decrease the surface area with each attempt. What dimensions give the minimum surface area? Describe the shape of this prism.

9. **Reflect** Summarize your findings. Describe any relationship you notice between the length, width, and height of a square-based prism with minimum surface area for a given volume.

10. Save your spreadsheet for future use.

Example 1 Cardboard Box Dimensions

a) The Pop-a-Lot popcorn company ships kernels of popcorn to movie theatres in large cardboard boxes with a volume of 500 000 cm^3. Determine the dimensions of the square-based prism box, to the nearest tenth of a centimetre, that will require the least amount of cardboard.



b) Find the amount of cardboard required to make this box, to the nearest tenth of a square metre. Describe any assumptions you have made.

Solution

a) A square-based prism with a given volume has minimum surface area when it is a cube.

The formula for the volume of a cube is $V = s^3$, where s is the length of a side of the cube.

Substitute the given volume of 500 000 cm^3. Find s.

$$500\,000 = s^3$$

$$\sqrt[3]{500\,000} = \sqrt[3]{s^3}$$

Take the cube root of both sides.

I want a number whose cube is 500 000.
? × ? × ? = 500 000

$$\sqrt[3]{500\,000} = s$$

C 500000 $\sqrt[3]{\ }$

$$79.4 \doteq s$$

For the cube root function, press MATH and select **4:$\sqrt[3]{\ }$(**.

To use the least amount of cardboard, the popcorn should be shipped in a cube-shaped box with side lengths of 79.4 cm.

Literacy Connections

A cube is a number that is the product of three identical factors. Each of the factors is the cube root of the number.

For example, the cube root of eight is two.

$2^3 = 2 \times 2 \times 2$
$= 8$

b) The amount of cardboard needed is the surface area of the box. A cube has six square faces.

$$\begin{aligned} SA &= 6s^2 \\ &= 6(79.4)^2 \\ &\doteq 38\,000 \end{aligned}$$

There are 100 cm in 1 m. So, there are 10 000 cm^2 in 1 m^2.

To express 38 000 cm^2 in square metres, divide by 100^2, or 10 000.

$$\frac{38\,000}{10000} = 3.8$$

It would take about 3.8 m^2 of cardboard to make this box. However, these calculations do not take into account the extra cardboard for the seams or any overlapping flaps.

Example 2 Minimize Heat Loss

Did You Know?

There are different types of insulators.

- Thermal insulators reduce the flow of heat.
- Electrical insulators reduce the flow of electricity.
- Acoustical insulators reduce the flow of sound.

Tyler has been asked to design an insulated square-based prism container to transport hot food. When hot food is placed in the container, it loses heat through the container's sides, top, and bottom. To keep heat loss to a minimum, the total surface area must be minimized.

a) Find the interior dimensions of the container with volume 145 000 cm^3 that has minimum heat loss. Round the dimensions to the nearest tenth of a centimetre.

b) What other factors might Tyler consider?

Solution

a) To minimize heat loss, Tyler must find the optimal surface area for a volume of 145 000 cm^3. The minimum surface area occurs when the container is cube-shaped.

Use the formula for the volume of a cube.

$$\begin{aligned} V &= s^3 \\ 145\,000 &= s^3 \\ \sqrt[3]{145\,000} &= s \\ 52.5 &\doteq s \end{aligned}$$

Take the cube root of both sides.

To minimize heat loss, the container should be cube-shaped with interior side lengths measuring about 52.5 cm.

b) Tyler may decide to design the container with a different shape, even though a cube would be the best for reducing heat loss. The container should also be shaped so that it is easy to carry, visually appealing, and handy to store, and holds the hot food conveniently.

Does a cube satisfy these criteria?

Key Concepts

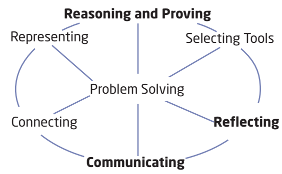
- Minimizing surface area for a given volume is important when designing packages and containers to save on materials and reduce heat loss.
- For a square-based prism with a given volume, a base length and a height exist that result in the minimum volume.
- For a square-based prism with a given volume, the minimum surface area occurs when the prism is a cube.
- Given a volume, you can find the dimensions of a square-based prism with minimum surface area by solving for s in the formula $V = s^3$, where V is the given volume and s is the length of a side of the cube.

Communicate Your Understanding

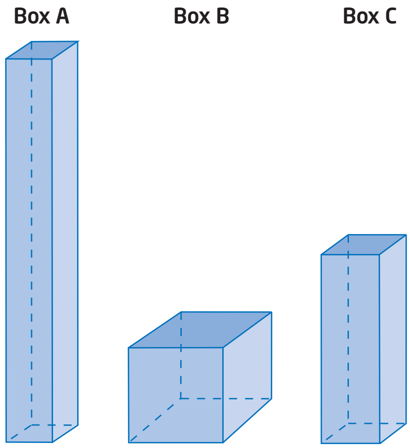
C1 Describe a situation when you would need to minimize the surface area for a given volume.

C2 If a cube-shaped box requires the least amount of material to make, why are all boxes not cube-shaped? Give an example of a situation where a cube-shaped box is not the most desirable shape.

Practise



1. These square-based prisms all have the same volume. Rank them in order from least to greatest surface area. Explain your reasoning.



For help with questions 2 and 3, see Example 1.

2. Determine the dimensions of the square-based prism box with each volume that requires the least material to make. Round the dimensions to the nearest tenth of a centimetre, when necessary.

a) 512 cm^3 **b)** 1000 cm^3 **c)** 750 cm^3 **d)** 1200 cm^3

3. Determine the surface area of each prism you found in question 2, to the nearest square centimetre.

For help with question 4, see Example 2.

4. Determine the dimensions of a square-based prism container with volume 3200 cm^3 and minimum heat loss. Round the dimensions to the nearest tenth of a centimetre.

Connect and Apply

5. Laundry detergent is packaged in a square-based prism box.

a) The box contains 4000 cm^3 of detergent. What dimensions for the box require the least amount of cardboard? Round the dimensions to the nearest tenth of a centimetre.

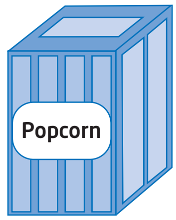
b) Does laundry detergent usually come in a box shaped like the one you found in part a)? Suggest reasons for this.

6. Chapter Problem Talia is shipping USB (universal serial bus) cables in a small cardboard squared-based prism box. The box must have a capacity of 750 cm^3 and Talia wants to use the minimum amount of cardboard when she ships the box.

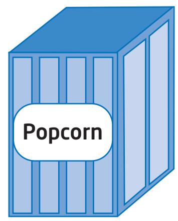
a) What should the dimensions of the box be, to the nearest hundredth of a centimetre?

b) What is the minimum amount of cardboard that Talia will need, to the nearest tenth of a square centimetre?

7. A movie theatre wants its large box of popcorn to be a square-based prism with a capacity of 2.5 L. Determine the least amount of cardboard required to construct this box, to the nearest square centimetre. Hint: 1 L = 1000 cm^3.



8. Refer to question 7. Usually, when you buy popcorn in a movie theatre, the box does not have a lid.



a) Carry out an investigation to determine the dimensions of a lidless box with minimum surface area and a capacity of 2.5 L.

b) Compare your results to those in question 7. Are the dimensions the same or different?

c) Does the lidless box require more, less, or the same amount of material to construct, compared to the box with a lid?

9. a) Determine the dimensions of a square-based prism juice box that holds 200 mL of juice and requires the least amount of material. Round the dimensions to the nearest tenth of a centimetre. Hint: 1 mL = 1 cm^3.

b) Suggest reasons why juice boxes are not usually manufactured with the dimensions you found in part a).

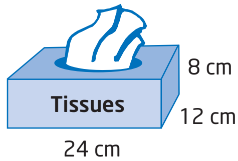
c) Write a letter to the manufacturer recommending a new design for its juice boxes, keeping your results from parts a) and b) in mind.

10. Create a problem that involves designing a square-based prism with minimum surface area. Solve the problem. Exchange with a classmate.

Extend

11. How would you arrange 100 interlocking cubes in a square-based prism with the smallest surface area possible?

12. A carton must be designed to hold 24 boxes of tissues. Each tissue box has dimensions 12 cm by 8 cm by 24 cm.



a) Design the carton so that it requires the least amount of cardboard. Include a diagram showing how the tissue boxes are to be stacked in the carton.

b) Explain why you think your design is the optimal shape.

c) Is packaging 24 boxes of tissue per carton the most economical use of cardboard? Explain your reasoning.

13. A warehouse is designed to provide 1000 m^3 of storage space. The surface area of the walls and roof must be kept to a minimum to minimize heat loss. Very little heat is lost through the floor, so you can ignore it. Carry out an investigation to determine the best dimensions for the warehouse.

14. **Math Contest** RiceCo ships long-grain rice in large cardboard boxes that hold 216 000 cm^3. Determine the least amount of cardboard needed for one of these boxes, if an extra 10% is used to join the sides of the boxes.

15. **Math Contest** At the Beautiful Box Company, cardboard boxes must hold 2700 cm^3. Determine the least amount of cardboard needed for one of these boxes if the boxes have right-triangular flaps to fasten the faces together. The triangular flaps are all the same size with one leg equal to one third the length of the box and the other leg equal to one third the width of the box.